KANTOWSKI-SACH ANISOTROPIC DARK ENERGY COSMOLOGICAL MODELS WITH TIME-DEPENDENT DECELERATION PARAMETER

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ABSTRACT

Kontowski-Sach cosmological models with variable equation of state (EoS) parameter have been investigated in general relativity when universe is filled with dark energy. The field equations have been solved by considering (i) q =B (variable), (ii) $R(t) = \left(t^n e^t\right)^{\frac{1}{l}}$ *which gives* $q = \frac{lr}{(t+r)^2} - 1$. *This renders early decelerating and late time accelerating cosmological models. The physical and geometrical properties of the models are also discussed.*

Keywords: *Dark Energy, Deceleration Parameter, Kontowski-Sach Space Time.*

1. INTRODUCTION

To study cosmological models, one of the important observational quantities is the deceleration parameter *q*. In any cosmological model, the Hubble constant *H* and deceleration parameter *q* play an important role in describing the nature of the evolution of the universe. The former one tells us the expansion rate of the universe today while the latter one characterizes the accelerating ($q < 0$) or decelerating ($q > 0$) nature of the universe. Some relativists assume various physical or mathematical conditions to obtain an exact solution of the Einstein's field equations. Berman [1][2] proposed a special law of variation for Hubble's parameter to obtain the cosmological solutions called the models with constant deceleration parameter (CDP) by assuming constant. This law is used by number of authors to study the cosmological models.

In 2006, Pradhan *et al.* [3] proposed the deceleration parameter be a variable parameter as: $q = -\frac{R\ddot{R}}{\dot{R}^2} = B(\text{var}\,iable)$, where *R* is the average scale factor. Yadav [4] and Chawla *et al.* [5] have studied cosmological models with variable deceleration parameter.

In 2012, Saha *et al.* [6] have obtained cosmological solutions for FRW universe filled with two fluids consisting of dark energy and barotropic fluid by selecting the average scale factor $R(t) = \sqrt{t^n e^t}$ which generates a time-dependent deceleration parameter such that the model generates a transition of the universe from early decelerating phase to the recent accelerating phase. Pradhan and Amirshachi [7] have also investigated accelerating dark energy models in Bianchi type-V space-time by selecting the scale factor as in Saha. However, Yadav [8][9], Pradhan[10], Rahman and Ansari [11] have generalized the average scale factor *a* given by $R(t)=(t^n e^t)^{\frac{1}{m}}$, where *n*, *m* are positive constants and obtained the cosmological solutions. Recently Pradhan *et al.* [10] and Rahman *et al.* [11] have studied cosmological models based on Lyra's geometry with constant and time-dependent displacement field in different context.

Motivated by this study about the deceleration parameter from constant to time dependent, an attempt is made to study Kantowski-Sach space-time when the universe is filled with DE with time-dependent DP in general relativity. This work is organized as follows: In Section 2, the model and field equations have been presented. The field equations have been solved in Section 3 by choosing two different time depending deceleration parameters. The physical and geometrical behaviors of the models have been discussed in Sections 4 to 5. In the last Section 6, concluding remarks have been expressed.

2.Metric and Field Equation

Kantowski-Sachs metric is considered in the form

$$ds^2 = dt^2 - a^2 dr^2 - b^2\left(d\theta^2 + \sin^2\theta d\psi^2\right) , \quad (1)$$

where $a(t)$ and $b(t)$ are scale factors and are functions of cosmic time t.

The energy-momentum tensor for a perfect fluid is

$$T_i^j = (\rho + p)u_i u^j - p g_i^j , \quad (2)$$

where p is the pressure, ρ is the energy density and g_j^i is a metric tensor. In co-moving coordinate system, u^i are the four co-moving velocity vectors which satisfy the condition

$$u^i u_i = 0 , \quad \text{for} \quad i = 1,2,3$$

and $$u^i u_i = 1 , \quad \text{for} \quad i = 0 .$$

From equation(2), the components of energy-momentum tensor are

$$T_0^0 = \rho , \qquad T_1^1 = T_2^2 = T_3^3 = -p . \quad (3)$$

With the help of equation(3), the matter tensor is given by

$$T_j^i = diag.(\rho, -p, -p, -p) . \quad (4)$$

For the perfect fluid, p and ρ are related by and equation of state

$$p = \omega\rho, \quad 0 \le \omega \le 1. \tag{5}$$

The Einstein's field equations are given by

$$R_i^j - \frac{1}{2} g_i^j R = -T_i^j, \tag{6}$$

where R_i^j is a Ricci tensor, R is the Ricci scalar.

The Ricci scalar for the Kantowski-Sachs metric is given by

$$R = 2\left(\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + 2\frac{\dot{a}\dot{b}}{ab} + \frac{1}{b^2}\right).$$

With the help of equations (4) and(5), the field equations(6), for the metric (1) are

$$2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} = \rho, \tag{7}$$

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{1}{b^2} = -\omega\rho, \tag{8}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = -\omega\rho. \tag{9}$$

Here the over dot $\left(\dot{}\right)$ represents the differentiation with respect to t.

3. SOLUTION OF THE FIELD EQUATIONS

The field equations (7) to (9) are a system of three non-linear differential equations with four unknown parameters. The system is thus initially undetermined. To obtain a deterministic solution the following physical conditions are used.

(i) The expansion scalar (θ) is proportional to shear (σ). This condition leads to

$$\frac{1}{\sqrt{3}}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) = \alpha_0\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right),$$

which yields

$$\frac{\dot{a}}{b} = m\frac{\dot{b}}{b},$$

Where α_0 and m are arbitrary constants.

Above equation after integration reduces to

$$a = \eta(b)^m,$$

where η is an integration constant.

Here, for simplicity and without loss generality, we assume that $\eta = 1$.

Hence, we have

$$a = b^m , \qquad m \neq 1 . \tag{10}$$

Collins *et al.* have pointed out that for the spatially homogeneous metric the normal congruence to the homogenous expansion satisfies that the condition $\frac{\sigma}{\theta}$ is constant [12].

(ii) Now, extra condition is needed to solve the system completely. Hence, different models of deceleration parameters are considered as discussed in the following sections .

4. MODEL WITH TIME DEPENDENT DECELERATION PARAMETER

The law of variation of Hubble parameter proposed by Saha *et al.* [6] modified by Singha and Debnath [13] called as special form of deceleration parameter defined by

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \frac{l\,r}{(t+r)^2} - 1 . \tag{11}$$

The average scale factor as an integrating function of time is (Saha et al. [6]) given by

$$R = (\mathrm{t}^r \mathrm{e}^t)^{\frac{1}{n}} . \tag{12}$$

For the metric(1), the average scale factor R is given by

$$R = \left(ab^2\right)^{\frac{1}{3}} . \tag{13}$$

From the equations (12) and(13),

$$R = \left(ab^2\right)^{\frac{1}{3}} = \left(t^r e^t\right)^{\frac{1}{l}}$$

$$\Rightarrow ab^2 = \left(t^r e^t\right)^{\frac{3}{l}} .$$

Using equation(10), it reduces to

$$b^m b^2 = \left(t^r e^t\right)^{\frac{3}{l}}$$

$$\Rightarrow b = \left(t^r e^t\right)^{\frac{3}{(m+2)l}} \quad . \tag{14}$$

With the help of equation(14), equation (10) leads to

$$a = \left(t^r e^t\right)^{\frac{3m}{(m+2)l}} \quad . \tag{15}$$

Thus, the Kantowski-Sach anisotropic DE cosmological model in general relativity with time-dependent deceleration parameter can be written as

$$ds^2 = dt^2 - (t^r e^t)^{\frac{6m}{(m+2)l}} dr^2 - (t^r e^t)^{\frac{6}{(m+2)l}} (d\theta^2 + \sin^2\theta d\psi^2) \; . \tag{16}$$

4.1 PHYSICAL PROPRTTIES OF THE MODEL

For the cosmological model(16), the physical quantities such as spatial volume V , Hubble parameter H , expansion scalar θ, mean anisotropy parameter A_m, shear scalar σ^2, energy density ρ and equation of state (EoS) parameter ω are obtained as follows:

The spatial volume is in the form

$$V = (t^r e^t)^{\frac{3}{l}} \; . \tag{17}$$

The rate of expansion H$_i$ are

$$H_x = \frac{3m}{(m+2)l}(rt^{-1}+1), \; H_y = \frac{3}{(m+2)l}(rt^{-1}+1) \, . \tag{18}$$

The Hubble parameter is given by

$$H = \frac{1}{l}(rt^{-1}+1) \; . \tag{19}$$

The expansion scalar is

$$\theta = 3H = \frac{3}{l}(rt^{-1}+1) \; . \tag{20}$$

The mean anisotropy parameter is

$$A_m = \frac{2\left(m-1\right)^2}{\left(m+2\right)^2} = \text{constant} \neq 0, \text{ for } m \neq 1 . \tag{21}$$

The shear scalar is given by

$$\sigma^2 = \frac{3(m-1)^2}{l^2(m+2)^2}(rt^{-1}+1)^2 \ . \tag{22}$$

It is observed that

$$\lim_{t\to\infty} \frac{\sigma^2}{\theta^2} = \frac{(m-1)^2}{3(m+2)^2} \neq 0 \ . \tag{23}$$

The energy density is

$$\rho = \frac{9(2m+1)}{(m+2)^2 l^2}(rt^{-1}+1)^2 + (t^r e^t)^{\frac{-6}{(m+2)l}} \ . \tag{24}$$

The EoS parameter is

$$\omega = -\frac{\left[\frac{27}{(m+2)^2 l^2}(rt^{-1}+1)^2 - \frac{6}{(m+2)l} rt^{-2} + (t^r e^t)^{\frac{-6}{(m+2)l}}\right]}{\left[\frac{9(2m+1)}{(m+2)^2 l^2}(rt^{-1}+1)^2 + (t^r e^t)^{\frac{-6}{(m+2)l}}\right]} . \tag{25}$$

For illustrative purposes, evolutionary behaviors of some cosmological parameters are shown graphically Fig. 1.

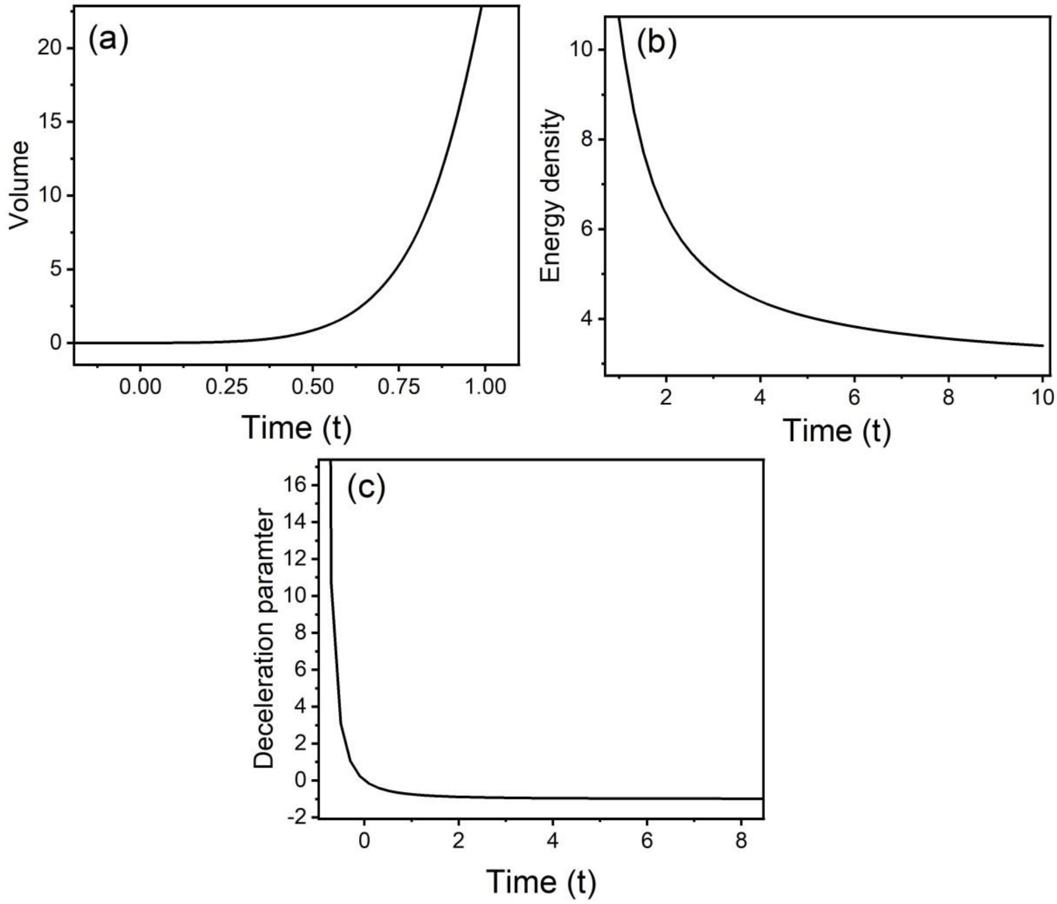

Figure 1 Plots of time versus (a) Spatial volume (b) Energy density (c) Decelaration parameter for the values $r=1$, $l=1$, $m=2$.

PHYSICAL BEHAVIOR OF THE MODEL

From figure 1 (a), we observed that the spatial volume is zero at $t=0$ and the expansion scalar is infinite showing that, the universe starts evolving with zero volume at $t=0$ and expands with cosmic time t which is big bang scenario. Also, scale factors are zero at the initial epoch $t=0$ hence the model has a point type singularity. At $t\to\infty$, we have $q=-1$ and $\frac{dH}{dt}=0$ indicating that the Hubble's parameter is maximum and the model has the fastest rate of expansion for $t\to\infty$. The mean anisotropy parameter A_m is constant and $\lim_{t\to\infty}\frac{\sigma^2}{\theta^2}\neq 0=const.$ is also constant, hence the model is anisotropic throughout the evolution of the universe except at $m=1$(*i.e.* the model does not approach isotropy). In figure 1 (b), the plot of energy density verses time is given which indicates that the model starts with infinite density and as time increases the energy density tends to a finite value. Hence, after some finite time, the model approaches steady state. In figure 1 (c), the plot of deceleration parameter verses time is given from which we conclude that the model is decelerating at an initial phase and changes from decelerating to accelerating. Hence the model is consistent with the recent cosmological observations (Perlmutter et al.[14], Garnavich et al. [15], Perlmutter et al. [16], Riess et al. [17], Schmidt et al. [18],Perlmutter et al. [19], Riess et al. [20]). Thus, our DE model is consistent with the results of recent observations.

5. MODELS WITH VARIABLE DECELERATION PARAMETER

The deceleration parameter is considered to be to be a variable parameter Pradhan *et al.* [3] as:

$$q=-\frac{R\ddot{R}}{\dot{R}^2}=B \quad \text{(variable)}, \tag{26}$$

where R is the average scale factor.

From equation(26),

$$\Rightarrow \frac{\ddot{R}}{R}+B\frac{\dot{R}^2}{R^2}=0 \ . \tag{27}$$

To solve equation(27), assume that $B=B(R)$. It is important to note here that one can assume $B=B(t)=B(R(t))$, as R is also a time-dependent function. It can be done only if there is one to one correspondence between t and R. However, this is only possible when one avoid singularity like big bang or big rip because t and R are increasing functions.

The general solution of (27) with assumption $B=B(R)$, is given by

$$\frac{\ddot{R}}{\dot{R}}=-\frac{B}{R}\dot{R} \ .$$

On integration with respect to t, the above equation simplifies to

$$\log \dot{R}=-\int\frac{B}{R}dR$$

$$\Rightarrow \dot{R}=e^{-\int\frac{B}{R}dR}$$

$$\Rightarrow \frac{\partial t}{\partial R}=e^{\int\frac{B}{R}dR} \ .$$

Again, on integration with respect to R

$$\int e^{\int\frac{B}{R}dR}dR=t+k \ , \tag{28}$$

where k is the constant of integration.

The term $\int e^{\frac{B}{R}} dR$ is chosen in such a manner that (28) is integrable. Hence, let

$$\int e^{\frac{B}{R}} dR = \ln L(R) \ , \tag{29}$$

$L(R)$ is a function of R only which does not affect the nature of generality of the solution.

From equations (28) and(29), we get

$$\int L(R)\, dR = t + k \ . \tag{30}$$

The choice of $L(R)$ in equation (30) is quite arbitrary but to get physically viable models of the universe consistent with observations, let us consider

$$L(R) = \frac{nR^{n-1}}{\alpha\sqrt{1+R^2}} \ , \tag{31}$$

where α is an arbitrary constant.

Without loss of generality assuming constants of integration to be zero.

On integration equation(30), it leads to

$$R = \left(\sinh(\alpha t)\right)^{\frac{1}{n}} \ . \tag{32}$$

From equations (32) and (13), we get

$$R = (ab^2)^{\frac{1}{3}} = (\sinh(\alpha t))^{\frac{1}{n}}$$

$$\Rightarrow ab^2 = (\sinh(\alpha t))^{\frac{3}{n}}$$

Using equation(10), it reduces to

$$b^m b^2 = (\sinh(\alpha t))^{\frac{3}{n}}$$

$$\Rightarrow \quad b = (\sinh(\alpha t))^{\frac{3}{(m+2)n}} \tag{33}$$

With the help of equation(33), equation (10) simplifies to

$$a = b^m = \left(\sinh(\alpha t)\right)^{\frac{3m}{(m+2)n}} \ . \tag{34}$$

Using equations (33) and(34), equation (1) leads to,

$$ds^2 = dt^2 - (\sinh(\alpha t))^{\frac{6m}{(m+2)n}} dr^2 - (\sinh(\alpha t))^{\frac{6}{(m+2)n}} (d\theta^2 + \sin^2\theta d\psi^2) \tag{35}$$

Equation (35) represents Kantowski- Sach anisotropic dark energy cosmological models with time dependent deceleration parameter.

5.1 PHYSICAL PROPERTIS OF THE MODEL

For the cosmological model (35), the physical quantities such as spatial volume V, Hubble parameter H, expansion scalar θ, mean anisotropy parameter A_m, shear scalar σ^2, energy density ρ, EoS parameter ω and deceleration parameter q are obtained as follows:

The spatial volume is in the form

$$V = (\sinh(\alpha t))^{\frac{3}{n}} . \tag{36}$$

The Hubble parameter is in the form

$$H = \frac{\alpha}{n}\coth(\alpha t) . \tag{37}$$

The expansion scalar is

$$\theta = 3H = 3\frac{\alpha}{n}\coth(\alpha t) . \tag{38}$$

The mean anisotropy parameter is in the form

$$A_m = \frac{2\left(m-1\right)^2}{\left(m+2\right)^2} = \text{constant} \neq 0, \text{ for } m \neq 1. \tag{39}$$

The shear scalar is

$$\sigma^2 = \frac{3\alpha^2(m-1)^2}{(m+2)^2 n^2}(\coth(\alpha t))^2 . \tag{40}$$

It is observed that

$$\lim_{t\to\infty}\frac{\sigma^2}{\theta^2} = \frac{(m-1)^2}{3(m+2)^2} \ \text{constant} \neq 0, \text{ for } m \neq 1 . \tag{41}$$

The energy density is

$$\rho = \frac{9\alpha^2(2m+1)}{(m+2)^2 n^2}(\coth(\alpha t))^2 + (\cos ech(\alpha t))^{\frac{6}{(m+2)n}} \tag{42}$$

The EoS parameter is

$$\omega = -\frac{\left[\frac{27\alpha^2}{(m+2)^2 n^2}(\coth(\alpha t))^2 - \frac{6\alpha^2}{(m+2)n}(\cos ech(\alpha t))^2 + (\cos ech(\alpha t))^{\frac{6}{(m+2)}}\right]}{\left[\frac{9\alpha^2(2m+1)}{(m+2)^2 n^2}(\coth(\alpha t))^2 + (\cos ech(\alpha t))^{\frac{6}{(m+2)n}}\right]} \tag{43}$$

For illustrative purposes, evolutionary behaviors of some cosmological parameters are shown graphically Fig. 2.

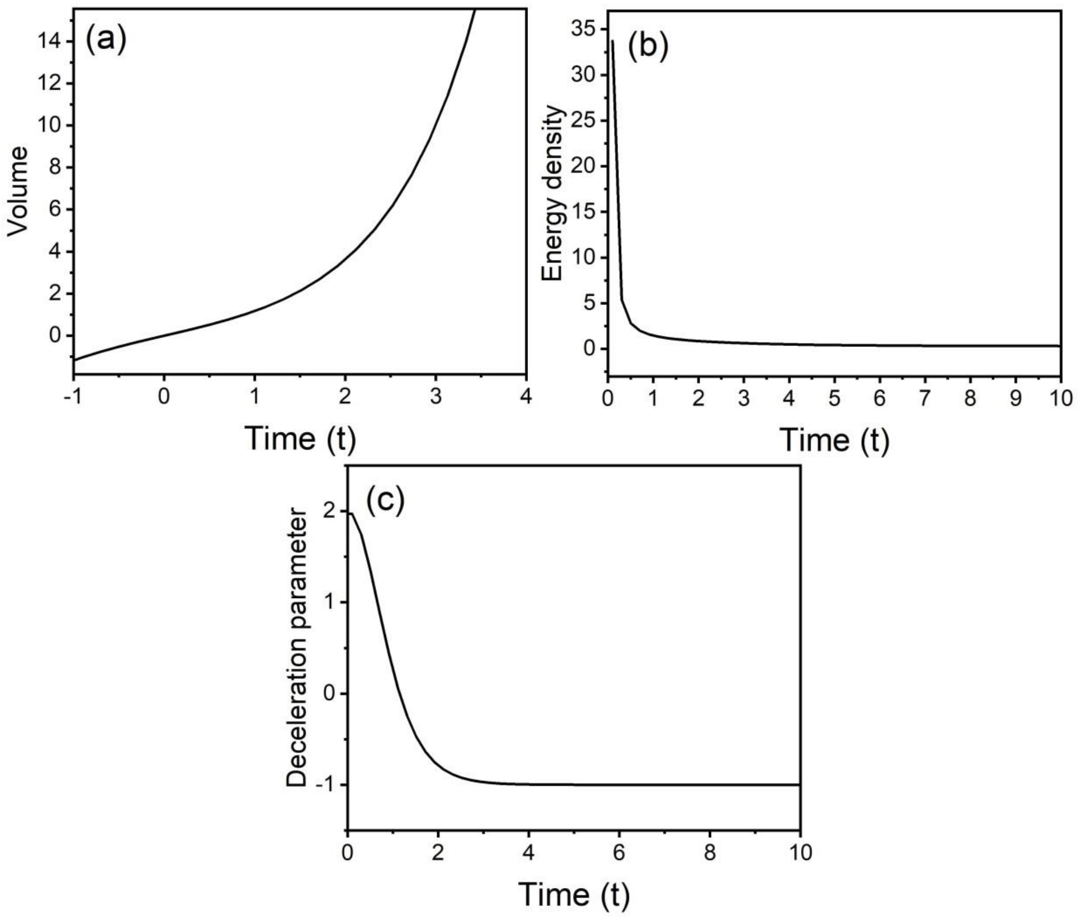

Figure 2 Plots of time versus (a) Spatial volume (b) Energy density (c) Decelaration parameter for the values $\alpha = 1$, $n = 3$, $m = 2$.

The deceleration parameter is

$$q = n\left(1 - \tanh^2(\alpha t)\right) - 1 \ . \tag{44}$$

PHYSICAL BEHAVIOR OF THE MODEL

From figure 2(a), we observed that the spatial volume is zero at $t = 0$ and the expansion scalar is infinite showing that, the universe starts evolving with zero volume at $t = 0$ and expands with cosmic time t which is big bang scenario. Also from equations (33) and (34), the spatial scale factors are zero at the initial epoch $t = 0$, hence the model has a point type singularity (MacCallum [21]). At $t \to \infty$, we have $q=-1$ and $\frac{dH}{dt}=0$ indicating that the Hubble's parameter is maximum and the model has the fastest rate of expansion for $t\to\infty$. From equation (39) and(41), the mean anisotropy parameter A_m is constant and $\lim_{t\to\infty}\frac{\sigma^2}{\theta^2}(\neq 0)$ is also constant, hence the model is anisotropic throughout the evolution of the universe except at $m=1$ *i.e.* the model does not approach isotropy. In figure 2(b), the plot of energy density versus time is represented which indicates that the model starts with infinite density and as time increases the energy density tends to a finite value. Hence, after some finite time, the model approaches steady state. In figure 2(c), the plot of deceleration parameter verses time is shown from which we can conclude that the model is decelerating at an initial phase and changes from the decelerating to accelerating phase. Hence, the model is consistent with the recent cosmological observations (Perlmutter et al.[14], Garnavich et al. [15], Perlmutter et al. [16], Riess et al. [17], Schmidt et al. [18],Perlmutter et al. [19], Riess et al. [20]). Thus, our DE model is consistent with the results of recent observations.

6. C0NCLUSION

A Kantowski- Sach cosmological model has been obtained when the universe is filled with DE in general relativity. To find a deterministic solution, we have considered two different models of deceleration parameter which yields time-dependent scale factors.

In section (4), the solution of the field equations has obtained by choosing the time-dependent DP $q=\frac{lr}{(t+r)^2}-1$ which yields time-dependent scale factor $R=(t^r e^t)^{\frac{1}{l}}$. It is observed that the model has point type singularity. In the early phase of the universe, the value of deceleration parameter is positive while as $t\to\infty$, the value of $q=-1$. Hence the universe had a decelerated expansion in the past and had accelerated expansion at present which is in good agreement with the recent observations of SN Ia.

In section (5), the solution of the field equations has obtained by choosing the variable DP $q=B(\text{var}\,iable)$ which yields time-dependent scale factor $R=(\sinh(\alpha t))^{\frac{1}{n}}$. The model has point type singularity. The model has decelerated expansion at early stages and accelerated expansion at present.

It is worth to mention that in all cases, the models obtained are expanding, sharing, non-rotating and does not approach isotropy for large t. Further, the models are anisotropic throughout the evolution. Thus, DE models are in good harmony with recent cosmological observations(Perlmutter et al.[14], Garnavich et al. [15], Perlmutter et al. [16], Riess et al. [17], Schmidt et al. [18],Perlmutter et al. [19], Riess et al. [20]). We hope that these models will be useful for a better understanding of dark energy in cosmology to study an inflationary behavior of the Universe.

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