Advanced Extrapolation Method

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Abstract
Extrapolation is one of the best and powerful methods used to improve the convergence of a sequence. In numerical analysis extrapolation is introduced in which it steadily reduces the computational effort by providing higher order accuracy with minimal computations. Multiple solutions with different step-size are required in Richardson and Romberg extrapolation technique. Here we derive an extremely powerful extrapolation method from single solution with fixed step size. This extrapolation scheme can achieve truncation error of order $h^i$ from $(i+1)$ points.

Keywords: Extrapolation, Richardson, Romberg, Numerical Method, Taylor series, Differential Equation, Convergence, Iteration, Discretization

1. Introduction

ADVANCED EXTRAPOLATION METHOD

One of the best known numerical extrapolation method which is introduced by Lewis Fry Richardson [2] in which we can accelerate the solution by reducing the grid spacing by half, one-fourth ($h, h/2, h/4, h/8$ etc). Here we solve the differential equation with each step sizes and extrapolate them to higher order accurate result by elimination higher order derivatives in Taylor Series[1].

Although this method reduces the iteration effort by reducing the matrix size, more matrices with different step size have to be used to obtain higher order approximation.

If we have $n$ grids with step size $h$, then we have to solve $n*n$ matrix. In case of simple tri-diagonal matrix total non-zero terms becomes $(3m - 2)$ terms.

General form for non zero term in tri-diagonal matrix can be seen as$(3 * 2^s m - 2)$. Where $s=0, 1, 2, 3…$

Extrapolation 1: By decreasing the step size by half increases the matrix by $(2m)*(2m)$ which is equal to $4m^2*m$ matrix or non-zero terms becomes $(6m-2)$

Extrapolation 2: By decreasing the step size to quarter increases the matrix by $(4m)*(4m)$ which is equal to $16m^2$ matrix and non-zero terms becomes$(3 * 2^4 m - 2)$
In the case of large m*m matrix (m>>1) it is impractical to solve them directly as they contain \( m^2 \) terms. To reduce computational effort we use tri-diagonal matrix with first order accuracy or Box scheme or centered difference method with second order accuracy and then apply extrapolation to increase accuracy or order of the solution. In comparison with non zero m*m matrix with tri-diagonal extrapolated system has a considerable amount of precision is lost.

### Table 1. Study on Known Results

<table>
<thead>
<tr>
<th>Extrapolation number</th>
<th>Non-zero m*m matrix</th>
<th>Order of truncation error in m*m matrix</th>
<th>Number of non-zero term in tri-diagonal matrix</th>
<th>Order of truncation error Forward Difference</th>
<th>Order of Truncation Error in Box Scheme or Centered Difference</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>m^2</td>
<td>O(h^m)</td>
<td>3 * 2^0 m - 2</td>
<td>O(h^1)</td>
<td>O(h^2)</td>
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<tr>
<td>2</td>
<td>m^2</td>
<td>O(h^m)</td>
<td>3 * 2^1 m - 2</td>
<td>O(h^2)</td>
<td>O(h^4)</td>
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<tr>
<td>3</td>
<td>m^2</td>
<td>O(h^m)</td>
<td>3 * 2^2 m - 2</td>
<td>O(h^3)</td>
<td>O(h^6)</td>
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<tr>
<td>4</td>
<td>m^2</td>
<td>O(h^m)</td>
<td>3 * 2^3 m - 2</td>
<td>O(h^4)</td>
<td>O(h^8)</td>
</tr>
<tr>
<td>n</td>
<td>m^2</td>
<td>O(h^m)</td>
<td>3 * 2^{(m-1)} m - 2</td>
<td>O(h^m)</td>
<td>O(h^{2m})</td>
</tr>
</tbody>
</table>

For example in m*m matrix, order of truncation error is \( O(h^m) \) with \( m^2 \) points.

Forward Difference scheme will take \( 3 * 2^{(m-1)} m - 2 \) points.

It can be seen that for all m>1:

\[
3 * 2^{(m-1)} m - 2 > m * m \tag{1a}
\]

Similarly Box scheme will take \( 3 * 2^{(m-1)} \frac{m}{2} - 2 \) points

It can be seen that for all m>6:

\[
3 * 2^{(m-1)} \frac{m}{2} - 2 > m * m \tag{1b}
\]

Equation (1a) and (1b) shows that for larger values of n with higher order accuracy the Tri-diagonal solver with present extrapolation method takes more points than an ordinary n*n matrix to meets its order of accuracy.

Multiple matrix results with different step size are also required. This leads in search for alternate Extrapolation Method.
2. Methodology

Here we see an efficient extrapolation method which can give \((n-1)\)th order accuracy with \(n\) grids from a single iterated value.

\[
\begin{align*}
  f_j & \quad f_{j+1} & \quad f_{j+2} & \quad f_{j+3} & \quad f_{j+4} & \quad \ldots & \quad f_{j+n-2} & \quad f_{j+n-1} & \quad f_{j+n} \\
  * & \quad * & \quad * & \quad * & \quad * & \quad * & \quad * & \quad * & \quad *
\end{align*}
\]

Each point has its own function and there derivatives and \(j\) represents local point under consideration.

Now we have

\[
f_j'|h = \frac{f_{j+1} - f_j}{h} \quad (2a)
\]

Well known First Order Newton’s Forward Difference Formulae.

Similarly,

\[
f_j'|2h = \frac{f_{j+2} - f_j}{2h} \quad (2b)
\]

On generalizing we may obtain:

\[
f_{j+k}|(n-k)h = \frac{f_{j+n} - f_{j+k}}{(n-k)h} \quad (3)
\]

This gives first order derivative or slope about \((j+k)\) with step size \((n-k)\) \(h\).

\(0 \leq k < n\), \(n = 1,2,3,\ldots\) \& \(k = 0,1,2,\ldots\)

Taking the averages of adjacent derivative terms deduce an important relationship.

\[
\begin{align*}
  \frac{f_j' + f_{j+1}'}{2} |h &= \frac{1}{2} \left[ \frac{f_{j+1} - f_j}{h} + \frac{f_{j+2} - f_{j+1}}{h} \right] = \frac{f_{j+2} - f_j}{2h} = f_j'|2h \quad (4a) \\
  \frac{f_j' + f_{j+1}' + f_{j+2}'}{3} |h &= \frac{1}{3} \left[ \frac{f_{j+1} - f_j}{h} + \frac{f_{j+2} - f_{j+1}}{h} + \frac{f_{j+3} - f_{j+2}}{h} \right] = \frac{f_{j+3} - f_j}{3h} = f_j'|3h \quad (4b)
\end{align*}
\]

On generalizing we may found that:

\[
\frac{1}{(n-k)} \sum_{i=k}^{n-1} f_i'|h = f_{j+k}|(n-k)h \quad (5)
\]

This gives the first order derivative about \(j+k\) with step size \(h\).

Equation (3) and (5) gives:
\[
\frac{1}{(n-k)} \sum_{i=k}^{n-1} f'_{j+i|h} = \frac{f_{j+n} - f_{j+k}}{(n-k)h} = f'_{j+k|(n-k)h}
\] (6)

For \( k = [0,n-1] \)
\[
\frac{1}{n} \sum_{i=0}^{n-1} f'_{j+i|h} = \frac{f_{j+n} - f_{j}}{nh} = f'_{j_nh}
\] (7)

From the above relation \( f'_{j+k|h} \) at each station can be found or inversely.

First order Newton’s Forward Difference Formulae can be found from, Taylor Series:

\[
f_{j+1} = f_j + hf_j' + \frac{h^2}{2!}f_j'' + \frac{h^3}{3!}f_j''' + \frac{h^4}{4!}f_j^{''''} + \frac{h^5}{5!}f_j^{'''''} + \ldots \] (8)

\[
f_{j+2} = f_j + (2h)f_j' + \frac{(2h)^2}{2!}f_j'' + \frac{(2h)^3}{3!}f_j''' + \frac{(2h)^4}{4!}f_j^{''''} + \frac{(2h)^5}{5!}f_j^{'''''} + \ldots \] (9)

\[
f_{j+3} = f_j + (3h)f_j' + \frac{(3h)^2}{2!}f_j'' + \frac{(3h)^3}{3!}f_j''' + \frac{(3h)^4}{4!}f_j^{''''} + \frac{(3h)^5}{5!}f_j^{'''''} + \ldots \] (10)

\[
f_{j+m} = f_j + (mh)f_j' + \frac{(mh)^2}{2!}f_j'' + \frac{(mh)^3}{3!}f_j''' + \frac{(mh)^4}{4!}f_j^{''''} + \frac{(mh)^5}{5!}f_j^{'''''} + \ldots \] (11)

Equations (7), (8), (9), (10) can be re-written as:

\[
f'_{jh} = \frac{f_{j+1} - f_j}{h} = f_j + \frac{h}{2!}f_j'' + \frac{h^2}{3!}f_j''' + \frac{h^3}{4!}f_j^{''''} + \frac{h^4}{5!}f_j^{'''''} + \ldots \] (8a)

\[
f'_{j2h} = \frac{f_{j+2} - f_j}{2h} = f_j + \frac{(2h)}{2!}f_j'' + \frac{(2h)^2}{3!}f_j''' + \frac{(2h)^3}{4!}f_j^{''''} + \frac{(2h)^4}{5!}f_j^{'''''} + \ldots \] (9a)

\[
f'_{j3h} = \frac{f_{j+3} - f_j}{3h} = f_j + \frac{(3h)}{2!}f_j'' + \frac{(3h)^2}{3!}f_j''' + \frac{(3h)^3}{4!}f_j^{''''} + \frac{(3h)^4}{5!}f_j^{'''''} + \ldots \] (10a)

\[
f'_{jmh} = \frac{f_{j+m} - f_j}{mh} = f_j + \frac{(mh)}{2!}f_j'' + \frac{(mh)^2}{3!}f_j''' + \frac{(mh)^3}{4!}f_j^{''''} + \frac{(mh)^4}{5!}f_j^{'''''} + \ldots \] (11a)

For \( j \) points we have \( j-1 \) equations or \( m \) equations. \( \{m=j-1\} \)

Now we can eliminate higher order derivative terms. The order of the solution depends on the number of points we choose.
For (m+1) points Leading Truncation Error Term can be found as:

\[ TE = A \frac{h^m}{(m+1)!} f_j^{m+1} \]  

(12a)

Or

\[ TE_{total} = \sum_{i=m}^{\infty} A_i \frac{h^i}{(i+1)!} f_j^{i+1} \]  

(12b)

Where, A is a constant which defines the convergence or divergence of the extrapolation.

- \( A < (i+1)! \) Convergence
- \( A > (i+1)! \) Divergence
- \( A = (i+1)! \) Depends on \( h^m \) and \( f_j^{m+1} \)

Equations (7), (8), (9), (10) can be tabularised as:

**Table 2. Forward Taylor Series Expansion about point j**

<table>
<thead>
<tr>
<th></th>
<th>( f_j^1 )</th>
<th>( f_j^2 )</th>
<th>( f_j^3 )</th>
<th>( f_j^4 )</th>
<th>( f_j^5 )</th>
<th>( f_j^{(1!)} )</th>
<th>( f_j^{(2!)} )</th>
<th>( f_j^{(3!)} )</th>
<th>( f_j^{(4!)} )</th>
<th>( f_j^{(5!)} )</th>
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</tbody>
</table>

Higher order derivative terms are to be eliminated.

**2.1 Elimination of higher order terms**

Higher order terms are eliminated using Gauss elimination.

**2.1.1 Elimination of second derivative terms**

It can be seen that the co-efficient term of \( f_j^{(2)} \) at row ‘m’ is \( m \).

**Elimination step:**

\[ E_m = \left( (m-1)^{th} \text{row} \right) \left( \frac{m}{m-1} \right) \left( (m)^{th} \text{row} \right) \]  

(13)

Value at \( (m-1)^{th} \text{row} = m - 1 \)

Value at \( (m)^{th} \text{row} = m \)

\[ E_m = (m-1) \left( \frac{m}{m-1} \right) - (m) \]

\[ E_m = 0 \]  

(14)
Equation (14) shows that step eliminated $f''_{(2!)}$ terms.

### 2.1.2 Elimination of third derivative terms

The co-efficient term of $f'''_{(3!)}$ at row ‘m’ was $m^2$.

First elimination altered its value

**Elimination:**

$$E_m = ((m - 1)\text{th row}) \left( \frac{m}{m - 1} \right) - ((m)\text{th row})$$

$$E_m = ((m - 1)^2) \left( \frac{m}{m - 1} \right) - ((m)^2)$$

$$E_m = (m - 1)(m) - (m^2)$$

$$E_m = -m$$

(15)

Interestingly after first elimination the co-efficient of $f'''_{(3!)}$ at row ‘m’ becomes $m$ with an inverted sign.

Steps used in 2.1 can be used to eliminate $f'''_{(3!)}$ terms.

### 2.1.3 Elimination of forth derivative terms

The co-efficient term of $f''''_{(4!)}$ at row ‘m’ was $m^3$.

First elimination altered its value

**1st Elimination:**

$$E_m = ((m - 1)\text{th row}) \left( \frac{m}{m - 1} \right) - ((m)\text{th row})$$

$$E_m = ((m - 1)^3) \left( \frac{m}{m - 1} \right) - ((m)^3)$$

$$E_m = m((m - 1)^2 - (m^2))$$

$$E_{m-1} = (m - 1)((m - 2)^2 - (m - 1)^2)$$

(16)

**2nd Elimination:**

$$E_m = ((m - 1)\text{th row}) \left( \frac{m}{m - 1} \right) - ((m)\text{th row})$$
\[ E_m = \left((m-1)(m-2)^2 - (m-1)^2\right)\left(\frac{m}{m-1}\right) \]
\[ - m((m-1)^2 - (m^2)) \]

\[ E_m = 2m \]

**3rd Elimination:**
\[ E_m = 2(m-1)\left(\frac{m}{m-1}\right) \]
\[ - 2(m) \]

\[ E_m = 0 \]

Similarly higher order terms can be eliminated.

### 2.2 Extrapolation co-efficients

Using elimination step we can trace the derivative terms \( f'_j, f'_{j2h}, f'_{j3h}, f'_{j4h}, f'_{j5h} \ldots \)

\( m=2 \)
\[ f'_E = \left(\frac{2}{1}\right) f'_{j} - f'_{j2h} \]  
(19a)

\( m=3 \)
\[ f'_E = \left(\frac{2}{1}\right) f'_{j} - f'_{j2h} \left(\frac{3}{2}\right) f'_{j2h} - f'_{j3h} \]  
(19b)

\( m=4 \)
\[ f'_E = \left(\frac{2}{1}\right) f'_{j} - f'_{j2h} \left(\frac{3}{2}\right) f'_{j2h} - f'_{j3h} \left(\frac{4}{3}\right) f'_{j3h} - f'_{j4h} \]  
(19c)

\( m=5 \)
\[ f'_E = \left(\frac{2}{1}\right) f'_{j} - f'_{j2h} \left(\frac{3}{2}\right) f'_{j2h} - f'_{j3h} \left(\frac{4}{3}\right) f'_{j3h} - f'_{j4h} \left(\frac{5}{4}\right) f'_{j4h} - f'_{j5h} \]  
(19d)
2.2.1 Co-efficient of $f'_{jh}$:

$$\frac{-m(m-1)(m-2)(m-3)(m-4)(m-5) * \ldots * 1}{(m-1)(m-2)(m-3)(m-4)(m-5) * \ldots * 1 * (1)} = -m$$  \hspace{1cm} (20a)

2.2.2 Co-efficient of $f'_{j2h}$:

$$\frac{m(m-1)(m-2)(m-3)(m-4)(m-5) * \ldots * 1}{(m-2)(m-3)(m-4)(m-5) * \ldots * 1 * (1 * 2)} = \frac{m(m-1)}{2}$$  \hspace{1cm} (20b)

2.2.3 Co-efficient of $f'_{j3h}$:

$$\frac{-m(m-1)(m-2)(m-3)(m-4)(m-5) * \ldots * 1}{(m-3)(m-4)(m-5) * \ldots * 1 * (1 * 2 * 3)} = -\frac{m(m-1)(m-2)}{6}$$  \hspace{1cm} (20c)

2.2.4 Co-efficient of $f'_{j4h}$:

$$\frac{m(m-1)(m-2)(m-3) * \ldots * 1}{(m-4)(m-5) * \ldots * 1 * (1 * 2 * 3 * 4)} = \frac{m(m-1)(m-2)(m-3)}{24}$$  \hspace{1cm} (20d)

2.2.5 Co-efficient of $f'_{j5h}$:

$$\frac{-m(m-1)(m-2)(m-3) * \ldots * 1}{(m-5)(m-6) * \ldots * 1 * (1 * 2 * 3 * 4 * 5)} = -\frac{m(m-1)(m-2)(m-3)}{120}$$  \hspace{1cm} (20e)

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Table 3. Solution for Forward Extrapolation

2.3 SOLUTION
A general formula can be deduced as
\[ f'_{VE} = \sum_{n=1}^{m} (-1)^{n+1} \frac{m!}{(m-n)!n!} f'_{jn}h \]  

(21)

Leading Truncation Error term:
\[ TE = \frac{(-1)^{m+1}}{(m+1)} h^m f_j^{(m+1)} \]  

(22a)

Total Truncation Error:
\[ TE_{total} = \sum_{i=1}^{\infty} (-1)^i \frac{m!}{(m-i)!i!} f_i^i h \]  

(22b)

Now we have a direct relationship (Equation 21) between Extrapolated value \( f'_{VE} \) and local first order derivatives with different step size \( f'_{jn}h \).

\[ f'_{VE} = \sum_{n=1}^{m} (-1)^{n+1} \frac{m!}{(m-n)!n!} f'_{jn}h \]

This can be written as:
\[ f'_{VE} = A_1 f'_{j1}h + A_2 f'_{j2}h + A_3 f'_{j3}h + A_4 f'_{j4}h + \ldots + A_m f'_{jmh} \]

(23)

Where \( A_1, A_2, A_3, A_4, \ldots, A_m \) are constants.

\[ A_n = (-1)^{n+1} \frac{m!}{(m-n)!n!} \]  

(24)

It’s already (Equation 5) known that:
\[ f'_{i=h} = \frac{f'_i|_{h} + f'_{i+1}|_{h} + f'_{i+2}|_{h} + \ldots + f'_{i+n-1}|_{h}}{n} \]

\[ f'_{VE} = A_1 f'_{j1}|_{h} + A_2 \frac{f'_{j2}|_{h} + f'_{j+1}|_{h}}{2} + \ldots + A_n \frac{f'_{jn}|_{h} + f'_{j+1}|_{h} + \ldots + f'_{j+n-1}|_{h}}{n} \]  

(25)

Coefficient of \( f'_{j}|_{h} \), \( A_1 + \frac{A_2}{2} + \ldots + \frac{A_m}{m} \)

Similarly for \( f'_{j+1}|_{h} \), \( A_2 + \frac{A_3}{2} + \ldots + \frac{A_m}{m} \)

More Generally for \( f'_{j+k} \), \( \sum_{n=k+1}^{m} A_k \frac{1}{n} \)

Or
\[ \sum_{n=k+1}^{m} (-1)^{n+1} \frac{m!}{(m-n)!n!} \frac{1}{n} \]  

(26)
\[ f_{VE}' = \sum_{k=0}^{m} \left[ \sum_{n=k+1}^{m} (-1)^{n+1} \frac{m!}{n(m-n)!n!} \right] f_{j+k}' \]

\[ TE = \frac{(-1)^{(m+1)} k^m f_j^{(m+1)}}{(m+1)} \]

*Truncation Error is not changed!

\( f_{VE}' \) Extrapolated value
\( m \) number equations or (i-1)
\( n \) local term number
\( h \) step size

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Now the extrapolation formula is in terms of local first order derivative value. We can choose any number of points according to necessity. Theoretically we may use any number of points with large order of accuracy. This may be limited by the precision error on the device.

3. Generalized Extrapolation Formulae

3.1 One Step Backward and Other Forward

Newton’s Forward Difference Formula is used in the initial stage. Newton’s Backward Difference formulae can also be used to obtain same results with inverted sign.

Using similar method we can deduce \((m-1)^{th}\) order solution for \(m\) backward points and \(m^+\) forward points.
In practical cases we may not always get enough forward or backward points to achieve desired order of accuracy. Here we may deal with the combination of forward and backward points.

\[ m = m^+ + m^- \] (29)

**Table 5. Taylor Series for One-Step Backward Extrapolation**

<table>
<thead>
<tr>
<th>( f'_{-jh} )</th>
<th>( f'_{jh} )</th>
<th>( f'_{j2h} )</th>
<th>( f'_{j3h} )</th>
<th>( f'_{j4h} )</th>
<th>( f'(1!) )</th>
<th>( f''(2!) )</th>
<th>( f'''(3!) )</th>
<th>( f''''(4!) )</th>
<th>( f'''''(5!) )</th>
<th>( f''''''(6!) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
<td>243</td>
<td>243</td>
<td>243</td>
<td>243</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
</tr>
</tbody>
</table>

Here, \( m^- = 1 \) and \( m^+ = 1,2,3, ... \)

Elimination method is described in the previous section.

Eliminating higher order derivative terms in each step implies:

**Table 6. Solution to One-Step Backward Extrapolation**

<table>
<thead>
<tr>
<th>( m )</th>
<th>( f'_{-jh} )</th>
<th>( f'_{jh} )</th>
<th>( f'_{j2h} )</th>
<th>( f'_{j3h} )</th>
<th>( f'_{j4h} )</th>
<th>( f'(1!) )</th>
<th>( f''(2!) )</th>
<th>( f'''(3!) )</th>
<th>( f''''(4!) )</th>
<th>( f'''''(5!) )</th>
<th>( f''''''(6!) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-1) ( \frac{1! \cdot 0!}{1!} )</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(-1) ( \frac{1! \cdot 1!}{2!} )</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(-1) ( \frac{1! \cdot 2!}{3!} )</td>
<td>-1</td>
<td>-3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(-1) ( \frac{1! \cdot 3!}{4!} )</td>
<td>-1</td>
<td>-6</td>
<td>4</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(-1) ( \frac{1! \cdot 4!}{5!} )</td>
<td>-1</td>
<td>-10</td>
<td>10</td>
<td>-5</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-24</td>
</tr>
</tbody>
</table>

A multiplier term exist, which was absent in forward extrapolation. The multiplier depends on \( m^- \) and \( m^+ \).
3.2 Two Step Backward and Other Forward

Table 7. Taylor Series for Two-Step Backward Extrapolation

<table>
<thead>
<tr>
<th>$f'_{-2jh}$</th>
<th>$f'_{-jh}$</th>
<th>$f'_{jh}$</th>
<th>$f'_{j2h}$</th>
<th>$f'_{j3h}$</th>
<th>$f''$ (1!)</th>
<th>$f'''$ (2!)</th>
<th>$f''''$ (3!)</th>
<th>$f'''''$ (4!)</th>
<th>$f''''''$ (5!)</th>
<th>$f''''''''$ (6!)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>4</td>
<td>-8</td>
<td>16</td>
<td>-32</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, $m^- = 2$ and $m^+ = 1, 2, 3, ...$

Elimination method is described in the previous section. Eliminating higher order derivative terms in each step implies:

Table 8. Solution to Two-Step Backward Extrapolation

<table>
<thead>
<tr>
<th>$m$</th>
<th>$f'_{-2jh}$</th>
<th>$f'_{-jh}$</th>
<th>$f'_{jh}$</th>
<th>$f'_{j2h}$</th>
<th>$f'_{j3h}$</th>
<th>$f''$ (1!)</th>
<th>$f'''$ (2!)</th>
<th>$f''''$ (3!)</th>
<th>$f'''''$ (4!)</th>
<th>$f''''''$ (5!)</th>
<th>$f''''''''$ (6!)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>(-1)^2 2! * 0! / 2!</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>-2</td>
<td>6</td>
<td>-14</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(-1)^2 2! * 1! / 3!</td>
<td>-1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-4</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(-1)^2 2! * 2! / 4!</td>
<td>-1</td>
<td>4</td>
<td>4</td>
<td>-1</td>
<td>1</td>
<td>-4</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(-1)^2 2! * 3! / 5!</td>
<td>-1</td>
<td>5</td>
<td>10</td>
<td>-5</td>
<td>1</td>
<td>1</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It can be seen that with equal number of points two step backward has least truncation term than pure forward extrapolation. More generally, least truncation co-efficient obtain when $m^- = m^+$
3.3 PASCAL’S TRIANGLE AND GENERAL SOLUTION

The solution shows a pattern similar to the well known Pascal’s triangle. The solution can be split up into multiplier term and triangular term. Their product gives the solution. This analogy will help in understanding the pattern of the solution.

Table 9. Relationship between the Pascal’s Triangle and the General Solution

<table>
<thead>
<tr>
<th>Multiplier Term</th>
<th>m↓</th>
<th>Triangular Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1)^{(m^-)})</td>
<td>a</td>
<td>b c d e f g h</td>
</tr>
<tr>
<td>((m^-)! * (m^+)!) * (m!)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>2 -1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>2 -3 1</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>4 -6 4 -1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>5 -10 10 -5 1</td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
<td>6 -15 20 -15 6 -1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>7 -21 35 -35 21 -7 1</td>
</tr>
</tbody>
</table>

The absolute value in the triangular terms resembles the Pascal’s Triangle. The row can be counted in terms of \(m \begin{array}{l} 1, m^- + m^+ + 1 \end{array}\). The local values of \(n\) and \(\bar{m}\) depend on \(m^-\). The \((m^- + 1)^{th}\) column in triangular term is trimmed. For example in forward extrapolation \(m^- = 0\) such that \(a^{th}\) column is trimmed and \(n\) starts from \(b\) as \(1\). Similarly for \(m^- = 3\), column \(d\) is trimmed such that \(n = \{a = 1, b = 2, c = 3, e = 4, f = 5 ... m^+\}\)

Coefficient Term = Multiplier Term * Triangular Term

**Case 1:** For \(0 \leq n \leq m^- - 1\)

\[C^- = -1^{(m^-)} * \frac{(m^-)! * (m^+)!}{m!} * (-1)^{(n+1)} \frac{m!}{n! * (m-n)!}\]  

(30a)

This can be reduced as:

\[C^- = (-1)^{(m^-)} * (-1)^{(n+1)} \frac{(m^-)! * (m^+)!}{n! * (m-n)!}\]  

(30b)

**Case 2:** For \(m^- + 1 \leq n \leq m\)
\[ C^+ = -1^{(m^-)} \frac{(m^-)! \* (m^+)!}{m!} \]
\[ * (-1)^{(n+1)} \frac{m!}{n! \* (m-n)!} \]  \hspace{1cm} (31a)

This can be reduced as:
\[ C^+ = (-1)^{(m^-)} \frac{(m^-)! \* (m^+)!}{n! \* (m-n)!} \]  \hspace{1cm} (31b)

\[ f'_{VE} = \sum_{n=0}^{m-1} C^- f'_{j(n-m^-)h} + \sum_{n=m+1}^{m} C^+ f'_{j(n-m^-)h} \]  \hspace{1cm} (32)

\[ f'_{VE} = \sum_{n=0}^{m-1} (-1)^{(n-m^-+1)} \frac{(m^-)! \* (m^+)!}{n! \* (m-n)!} f'_{j(n-m^-)h} \]
\[ + \sum_{n=m+1}^{m} (-1)^{(n-m^-+1)} \frac{(m^-)! \* (m^+)!}{n! \* (m-n)!} f'_{j(n-m^-)h} \]  \hspace{1cm} (33)

**TRUNCATION ERROR:**
\[ TE = (-1)^{(m^+)} \frac{(m^-)! \* (m^+)!}{(m+1)!} h^m f_j^{(m+1)} \]  \hspace{1cm} (34)

This can be derived by putting \( n = m+1 \)

### 3.4 GENERALISED EXTRAPOLATION FORMULAE

**Backward Points:**
\[ f'_{VE}^- = \sum_{k=0}^{m^-} \sum_{n=k}^{m-1} (-1)^{(n-m^-+1)} \frac{(m^-)! \* (m^+)!}{(m^- - n)! \* (m-n)!} f'_{j-k} \]  \hspace{1cm} (35)

**Forward Points:**
\[ f'_{VE}^+ = \sum_{k=0}^{m^+} \sum_{n=m-k+1}^{m} (-1)^{(n-m^-+1)} \frac{(m^-)! \* (m^+)!}{(n-m^-)! \* n! \* (m-n)!} f'_{j+k} \]  \hspace{1cm} (36)

We knew that
\[ f'_{VE} = f'_{VE}^+ + f'_{VE}^- \]

Generalised extrapolation Formulae can be given by:
\[ f'_{VE} = \sum_{k=0}^{m-1} \sum_{n=k}^{m-1} (-1)^{(n-m+1)} \frac{(m^-)! \cdot (m^+)!}{(m^- - n)! \cdot (m - n)!} f'_{j-k} \]
\[ + \sum_{k=0}^{m^-} \sum_{n=0}^{n^-} (-1)^{(n-m+1)} \frac{(m^-)! \cdot (m^+)!}{(n - m^-) \cdot n! \cdot (m - n)!} f'_{j+k} \] (37)

**TRUNCATION ERROR:**

\[ TE = (-1)^{(m^+1)} \cdot \frac{(m^-)! \cdot (m^+)!}{(m+1)!} \cdot h^m f^{(m+1)} \] (38)

### Table 10. Comparison with older methods

<table>
<thead>
<tr>
<th>Extrapolation number</th>
<th>Non-zero m*m matrix</th>
<th>Order of truncation error in n*n matrix</th>
<th>Number of non-zero term in tri-diagonal matrix</th>
<th>Order of truncation error Forward Difference</th>
<th>Order of Truncation Error in Box Scheme or Centered Difference</th>
<th>Advanced Extrapolation Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>m²</td>
<td>O(h^n)</td>
<td>3 \cdot 2^{0}m - 2</td>
<td>O(h^{1})</td>
<td>O(h^{2})</td>
<td>O(h^{m})</td>
</tr>
<tr>
<td>2</td>
<td>m²</td>
<td>O(h^n)</td>
<td>3 \cdot 2^{1}m - 2</td>
<td>O(h^{2})</td>
<td>O(h^{4})</td>
<td>O(h^{2m})</td>
</tr>
<tr>
<td>3</td>
<td>m²</td>
<td>O(h^n)</td>
<td>3 \cdot 2^{2}m - 2</td>
<td>O(h^{3})</td>
<td>O(h^{6})</td>
<td>O(h^{3m})</td>
</tr>
<tr>
<td>4</td>
<td>m²</td>
<td>O(h^n)</td>
<td>3 \cdot 2^{3}m - 2</td>
<td>O(h^{4})</td>
<td>O(h^{8})</td>
<td>O(h^{4m})</td>
</tr>
<tr>
<td>n</td>
<td>m²</td>
<td>O(h^n)</td>
<td>3 \cdot 2^{(n-1)m} m - 2</td>
<td>O(h^{m})</td>
<td>O(h^{2m})</td>
<td>O(h^{m^2})</td>
</tr>
</tbody>
</table>

It can be seen from the table that even at the first extrapolation the order of error in advanced extrapolation method matches with large m*m matrix with m² terms. (3m-2) points were enough to obtain same order result. (3m-2) << m² for large m.
This clearly indicates the outrageous application of method.

### 4. Result

An extrapolation formula is derived from Taylor series. From i points i^{th} order accurate results are obtained which is equivalent to solving i*i matrix or i² non negative terms. This problem is reduced to tri-diagonal matrix with (3i-2) non-negative terms.
Acknowledgments

Words are not enough to thank Dr. Raman Balu Dean (Research) Aeronautical Engineering, ACE College of Engineering for his proper guidance and encouragement.

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References

11.1. Journal Article
