Fixed Point Theory in b-Metric Space: A Review

Rubal Nandal

Department of Mathematics, Chandigarh University, Gharuan(Punjab)
E-mail: rubalnandal23@gmail.com,

Abstract

In the present paper we tend to discuss regarding the various stages of fixed point theory in b-metric space. We also discuss about the different types of mappings in b-metric space like self mapping, multivalued contractive mapping and compatible mapping of type(A) for t. We also tend to discuss about a unique fixed point in b-metric space for Kannan type or Chatterjea type contractive mappings.

Keywords: Metric space, Contractive mapping, Fixed point, b-metric space.

Introduction

Poincare[1] started working on Fixed point theory in 1886. It is a very important tool to determine the existence and uniqueness of the solution. It has many applications in engineering, computer sciences, non-linear analysis or many other branches of science like physics, chemistry and biology. In b-metric space there may be one fixed point or more than one for different kinds of mappings.

In 1906, Maurice Frechet discussed the idea of metric space in his work. In topology Brouwer[2] proved a fixed point theorem in 1912. Generally, metric space is a distance function, which finds the distance among all the members of a set. In fixed point theory the most important and widely used theorem is Banach Contraction Principle that was verified by the Stefan Banach[3] in 1922. According to this theorem, let (P,d) be a metric space where $P \neq \Phi$ and let A is a function with mapping $A: P \to P$ is known as the contraction if their exist $\beta > 0$ such that

$$\mathbf{d(Au, Av) \leq \beta\, d(u, v)}$$

Here. β is a lipschitz constant which is less than 1.

Further, Kannan[4] extended the theorem Banach contraction theorem i.e $A:Y \to Y$

$$\mathbf{d(Av, Av) \leq \alpha[d(u, Au)+d(v, Av)]} \qquad \alpha \in [0,1/2]$$

Chatterjea[5] introduced new type of contractive type mapping s.t.

$$\mathbf{d(Au,Av) \leq \alpha[d(u,Av)+d(v,Au)]} \qquad \alpha \in [0,1/2]$$

Firstly, in 1989 the b-metric space idea was mentioned by the Bakhtin [6], that ends up into the generalization of metric space. First of all the b-metric space appeared with in the work of Bakhtin [6] and Czerwik [7]. In b-metric space, we take a real no. with in the triangular inequality condition. Normally, the generalization of usual metric space is the b-metric space. On the motivation of b-metric space and G-metric space we have a tendency to generalize the b-metric space and the existence & uniqueness of fixed point for multivalued contraction mapping proved by the help of Picard and Jungck iterations. Later many researchers used the Banach Contraction Principle and generalized the result by using b-metric space. The multivalued contraction mapping initiated by Nadler[8] in 1969 for fixed point. There is also an important result of fixed point theory using compatible mapping which was introduced by Gerald Jungck[9] in 1986.

After that many types of compatible mapping introduced by many other authors. One of the compatible mapping is type(A) which was introduced in Jungck et al.[10] in 1993. There are many application in metric space as integral equation or non linear equations in which b-metric is used and the b-metric space also used in the iterations like picard's and

Jungck. Now we review regarding the proof of fixed point theorem by existence & uniqueness in b-metric space and also review about the fixed point theorem by compatible mapping of type(A) for two self mappings using the contractive condition or b-metric space for multivalued contractive mapping.

Main Result

In present review paper we discuss about the development of fixed point theory in b-metric space. Here we also discuss about the different type of mappings and extensions of b-metric space.

Czerwik[7] in his paper "Contraction mapping in b-metric space" proved the extension of Banach fixed point theorem with in the b-metric space .

Theorem1 Let $Y \neq \Phi$ and (Y, d) is a complete b-metric space and let A is a function with mapping $P : Y \to Y$ satisfies

$$d[P(u), P(v)] \leq \psi [d(u, v)], \quad \forall\ u,v \in Y,$$

where if ψ: R+$\to$ R+ is an increasing function, we have

$$\lim_{n\to\infty} \Psi^n (p) = 0$$

or every fixed point greater than zero. Then w is the only one fixed point for P and

$$\lim_{n\to\infty} d[A^n(m), w] = 0$$

In above theorem author proved the Banach fixed point theorem in b-metric space. Here firstly we take a set of natural number and there is an increasing function ψ then there is a contraction condition after that there is a sequence which is Cauchy sequence which converges uniformly. Then we get a fixed point by the continuity. We can also get a fixed point for the decreasing function also.

Maria et al.[13] in his paper "Fixed point theorems on multivalued mappings in b-metric space" proved the common fixed point theorem for multivalued mappings in complete b-metic space.

Theorem2 Let a function A of mapping $A : Y \to CS(Y)$ be a multi valued generalized contraction mapping where (Y,d) is a complete b-metric space where $Y \neq \Phi$ with $b \geq 1$ where b is constant. Then A has a fixed point**.**

In above theorem author proved the b-metric space for multivalued mapping . Here we take some points and than obtain a sequence which is Cauchy sequence than the sequence $\{x_n\}$ is convergent so the (Y,A) be the complete b-metric space . Then by existence and Uniqueness we get a has unique fixed point.

There are many other theorems related to this.

Mehmet Kir and Hukmi Kiziltunc[14] in his paper "On some well known fixed point theorems in b-metric spaces" Proved the Chatterjea and Kannan type contractive mappings which has a unique fixed point in b-metric space.

Theorem3 Let $Y \neq \Phi$ and let (Y, d) is a Complete b-metric space with constant b which is $b \geq 1$ and define the $\{y_n\}_{n=1}^{\infty}$ contain in k by recursion. Let P be a function of mapping from Y to Y ($P : Y \to Y$) be a mapping for which there exist $\alpha \in (0,1/2)$ such that,

$$d (Pu\ Pv) \leq \alpha[d(u, Pu)+d(v, Pv)] \quad \forall\ u, v \in Y .$$

Then, there exist $u^* \in Y$ such that $u_n \to u^*$ and u^* is the unique fixed point of P. This is the Kannan fixed point theorem in b-metric space.

Here author proved the b-metric space for kannan type fixed point theorem. Here first we take a sequence and then by contraction mapping the sequence is Cauchy sequence and it is convergent then by uniqueness P has a unique fixed point.

Theorem 4 Let (Y,d) is a complete b-metric space where Y≠Φ & define the $\{y_n\}_{n=1}^{\infty}$ sequence contain in Y by recursion. Let A:Y→Y be a mapping under the terms bα ∈ [0, ½) such that

$d(Au,Av) \leq \alpha\, [d(u,AV)+d(v,AU)]$ for all u,v ∈ Y , Then

their exist $u^* \in Y$ such that $u_n \to u^*$ and the unique fixed point of A is u^* . This is the chatterjea type fixed point theorem in b-metric space.

Here author proved the Chatterjea theorem also have a fixed point by taking a contraction condition then sequence is Cauchy sequence which is convergent. Then by uniqueness we get A has a fixed point.

Agarwal et al.[15] in his paper "Fixed point theorem in b-metric space" proved the completeness and uniqueness of fixed point theorem in b-metric also they proved the different type of contractive mappings.

Theorem 5 Let (Y, d) is a complete metric space and Y≠Φ. Let A is a function of mapping A:Y→Ysuch that,

$d(Ar, As) \leq a \max\{d(r, Ar), d(s, As), d(r, s)\} + b\{d(r, As)+d(s, Ar)\}$

where a, b are greater than zero such that, a+2bs ≤ 1 for all r, s ∈ Y and s ≥ 1 then A has a fixed point.

Here author proved the theorem by taking sequence and than there are three case arises, there is a convergent cauchy sequence than the (Y,d) is a complete metric space after uniqueness we get A has a unique fixed point x^* .

U. Karuppiah and M. Gunaseelan[16] in his paper "Fixed point theorem for multivalued contractive mappings in b-metric space" proved the-b-metric space for multivalued contractive mappings in fixed point theorem.

Theorem6 Let Y is not a empty set than (Y,d) be a complete b-metric space with constant b≥1 and the mapping S:Y→ CS(Y) be multivalued map satisfying

$H(Su,Sv) \leq md(u,v)+n \max\{d(u,Su),d(v,Sv)\}+o[d(u,Sv)+d(v,Sv)]$ ∀ u,v ∈ Y and m,n,o ∈ [0,1) are constant such that m+n+o < 1. Then S has one fixed point in Y.

Here the author generalized the agarawal et al. theorem[15]. In this theorem author proved the multivalued contractive mappings in b-metric space. we take some points and than take contractive condition. After that, there are two cases arises than by taking a sequence which is Cauchy and convergent so (Y,d) is a complete metric space and by uniqueness we get S has a fixed point in Y.

In support of the above theorem, there is also an example

Example: Let Y=[0,1] than a function A of mapping A:Y×Y→ Y by

$d(u, v)=|u - v|^2$ for all u, v ∈ Y

Then Y is non empty set &(Y,d) is a complete metric space.

Let A:Y×Y→ CS(Y) by Tu= u/7 for all u,v ∈ Y. Then,

H(Au,Av) = 1/49d(u,v)

(where b=c=0, a=1/49)

Therefore, 0 ∈ y is a fixed point.

Sanodia et al.[17] in his paper "Common fixed point theorem in b-metric space compatible mapping of type(A)" proved the fixed point theorem for compatible mapping of type(A) in b-metric space for two self mapping using contractive conditions.

Therorem7 Let Y≠Φ and let (Y,d) is a complete b-metric space with the b ≥1 where b is constant and let P and Q be two self mapping such that

i. Q(Y) ⊆ P(Y)
ii. Out of P and Q one will be continuous.
iii. (P,Q) is compatible mapping of type(A).
iv. $d(Qu,Qv) \leq m \max\{d(Qu,Pv)\ d(Qu,Qv),d(Qv,Pv),d(Qu,Pu)\}+n\{d(Qv,Pu)\}$

where m+2b ≤ 1 for all u, v ∈ Y then P and Q has a unique common fixed point. Here authors proves the fixed point for compatible mapping of type(A)in b-metric space.In this theorem there are two self mappings P and Q. Then by contractive conditions there are two cases and shows a convergent cauchy sequence than (Y,d) is a complete b- metric space. Than P is continuous and P,Q are the compatible of type(A). So, by contradiction we get P & Q has a fixed point and by the uniqueness we get a common fixed point.

Conclusion

There are many researchers who works on the b-metric space. Here, we tend to discussed regarding the b-metric space that is the generalization of usual metric space in fixed point theorem. Here we tend to review all the extensions, generalization of b-metric from Banach contraction, Kannan type, chatterjea type and many other theorems. We can say that the fixed point theory has many applications using b-metric space in single or multivalued mappings. In proof of all the theorems there is a convergent cauchy sequence and the space is complete b-metric space and than by uniqueness we get the unique fixed point.

The main point is that in every mapping like single valued, two self mapping, three self mapping and for multivalued mapping of b-metric we get the a fixed point. We can also extend the common fixed point theorem in b-metric space for compatible mapping of type(A) for the multivalued mapping using contractive conditions.

References

[1] Poincare H. (1886), Surless courbes define barles equations differentiate less, J.de Math., 2, pp. 54-65.
[2] Brouwer L. (1912), Uber Abbildungen von Mannigfaltigkeiten, Math. Ann., 70, pp. 97-115.
[3] Banach S. (1922), Sur les operations dans les ensembles abstracts ET leur applications aux equations integrals, Fund. Math.3, pp. 133-181.
[4] Kannan R. (1968), some results on fixed points, Bull. Calcutta Math. 60, pp.71-78.
[5] Chatterjea, S.K, Fixed point theorems, C.R.Acad. Bulgare Sci. 25, 1972, 727-730.
[6] Bakhtin, I. A., The contraction mapping principle in almost metric spaces, Funct. Anal., 30, Unianowsk, Gos. Ped. Inst., (1989), 26-37.
[7] Czerwik, S., Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis 1 (1993), 5-11.
[8] Nadler SB (1969) Multi-valued contraction mappings. Pac J Math 30:475–488.
[9] Jungck G., Compatible maps and common fixed points, Inter. J. Math. and Math. Sci. 9, 1986 771-779.
[10] Jungck G., Murthy P.P. and Cho Y.J., “Compatible mappings of type (A) and common fixed points”, Math. Japonica 38, 381- 390, 1993.
[11] Boriceanu, M., Fixed point theory for multivalued generalized contraction on a set with two b-metric, studia, univ Babes, Bolya: Math, Liv(3) (2009), 1-14.
[12] P. P. Murthy, Remarks on fixed point theorem of Sharma and Sahu, Bull. of Pure and Appl. Sc., 12 E(1-2), 1993 7-10.
[13] J. Maria Joseph, D. Dayana Roselin and M. Marudai,”Fixed point theorem on multivalued mapping in b-metric”,Maria Joseph et al. springerplus,2016,vol 5,pp.8.
[14] Kir Mehmet, Kiziltune Hukmi,”On some well known fixed point theorem in b-metric space”,Turkish Journal of analysis and number theory,2013,vol.1,no.13-16.
[15] S. Agrawal, K. Qureshi and Jyoti Nema,” A fixed point theorem for b-metric” Int.J. of Pure and applied mathematical sciences, volume 9,number1(2016),pp.45-50.
[16] U.Karuppiah and M.Gunaseelan,”Fixed point theorem for multivalued contractive mapping in b-metric”, Int. J.of mathematics trends and technology,2016,vol 37,number 4, pp.253-259.
[17] P.L. Sanodia, K.Qureshi, Swati Agarwal and Jyoti Nema,”Common fixed point theorem in b-metric space for compatible mapping of type(A)”,Int. J. of Mathematics Trends and Technology,2016.vol 35, no. 1,pp.15-18.