A GENERAL REVIEW ON GAME THEORY

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ABSTRACT
Game Theory can be applied to the field of Biology, especially evolution. In the early seventies, Game Theory, which was centered on the concept of a rational individual, was modified and enriched to be applied to a wide range of biological problems. In classical Game Theory, individuals could choose their strategies out of a certain set and could change them in repeated games. However, evolutionary Game Theory deals with entire populations whose members have fixed strategies - for instance, a type of behavior. A change of strategy is not a decision of a certain player but the replacement of certain individuals by their offspring. In this kind of games, what a player does depends on what everybody else is doing and that's why they are called frequency dependent games. Here it is not one species evolving against another, but the members of a certain species evolving against one another.

KEYWORDS-Game Theory, strategy, frequency.

INTRODUCTION

A game, in Game Theory, is a tool that can model any situation in which there are people that interact - taking decisions, making moves, etc - in order to attain a certain goal. This mathematical description of conflicts began in the twentieth century thanks to the work of John Von Neumann, Oskar Morgenstern and John Nash and one of its first motivations was to help military officers design optimal war strategies. Nowadays, however, Game Theory is applied to a wide range of disciplines, like Biology or Political Science, but above all, to Economy. Interestingly, eleven game-theorists have won the Economics Nobel Prize up to date but never has a Fields Medal been awarded to an expert in this field. This shows to what great extent Game Theory is important for Economy and at the same time how mathematicians regard it as a secondary discipline compared to other areas of Mathematics. This undergraduate thesis clearly falls under the category of applied mathematics or mathematical modeling and therefore its goal is far from just accurately proving a series of theorems. Instead, even if the foundations of Game Theory will be laid, I will focus on showing how Game Theory can be applied to solve a great number of different problems, like, for example, the emergence of cooperative dispositions towards strangers. Bearing this in mind, I will begin this undergraduate thesis by analyzing a military conflict between two countries whose officials will have a symbolic name: Nash and Neumann. To so do, I warn the reader that I will informally explain and use certain results that will be accurately justified later in this paper

A First Example

Suppose that a country A and a country B are at war and that the generals of each army are called, respectively, John Nash and John Von Neumann. Every single day, Nash will send a heavily armed bomber and a smaller support plane to attack B. To do so, he will put a bomb in one of the two aircrafts. At the same time, Neumann knows Nash's intentions and decides to embark on military action but judges unnecessary to attack the two planes, mainly for economic reasons. The bomber will survive 60% of the times it suffers an attack and, if it manages to live through the raid, it will always hit the target. The lighter plane, which is not as precise, hits the target 70% of the times and plus only survives Neumann's attack half of the times. There are clearly only four possible results, which come from the combination of Nash choice to put the bomb either in the bomber or in the support plane, and Neumann's call to attack one or the other aircraft. Nash feels that he gains when he hits the target and he does not care about suffering an attack on one of his planes. At the same time, Neumann desperately wants to protect his citizens and therefore he will lose utility when an attack is carried out. Nash's gain for every possible combination of strtegies is

	Bomber attacked	Support attacked
Bomb in bomber	0.6 × 1 = 0.6	1
Bomb in support	0	0, 5 × 0.7 = 0.35

Bomber attacked Support attacked Bomb in bomber 0.6 × 1 = 0.6 1 Bomb in support 0.7 0, 5 × 0.7 = 0.35 Table 1.1. Nash's Payoff where we have assumed an increase in a unit of his utility comes from a successful attack. In an isolated realization of the game, the target can only either be or not be hit, so what do the numbers in the matrix represent, Clearly, the expected values of the outcome of the game when those strategies are employed. Given that this game is repeated every day, the Law of Large Numbers guarantees that the average outcome of the confrontation of two strategies (let's say for example "Bomb in bomber" against "Bomber attacked") will tend to its expected value (for the aforementioned strategies, 0.6). Therefore, the following will be a good long-term analysis. Bear in mind that we should give the utility of both players, but here Neumann's utility has implicitly been given as it will be the matrix on top with a minus sign in front of all entries.

These facts could determine their strategies. If that was the case, the game would unfold as a constant 0.6 gain for Nash. However, instead of always putting the bomb in the bomber, Nash decides, every now and then, to bluff and to put the bomb in the support plane. How often should he do that Which is the best percentage of success he can get? How should Neumann adapt to this change? Let's give an answer to all this questions. Given that Nash will no longer stick to one of the strategies but combine them, he will start to employ a so-called mixed strategy. Let's use a probability distribution X = (a, b)to encapsulate this, where it should be understood that Nash will put the bomb in the bomber with a probability a and in the support plane with a probability b. Using this notation, a pure strategy is then (1, 0) -bomb always in the bomber- or

(0, 1) -bomb always in the support plane. Let us denote by A the matrix of table 1.1. For any strategies $X \in Sb_1$ and $Y \in Sb_2$ (Sbi should be understood as the set of all possible mixed strategies of player i) we can calculate the expected payoff H(X, Y) by moderating the payoffs of the pure strategies, i.e., by calculating XAY^T . For any strategy Nash (Neumann) picks, the strategy that minimizes the other's payoff is called the optimal counterstrategy or the best reply.

Theorem 1- If either Nash or Neumann employ a fixed strategy (and it means they will not keep changing the probabilities of how they distribute their choices), then the opponent's best reply is a pure strategy.

Proof - Assume Nash's strategy is $X = (1 - x, x)$ and Neumann's $Y = (1 - y, y)$. The expected payoff is the averaged payoff of every situation. For the sake of generality, let's assume a general (aij) payoff 2 × 2-matrix for Nash. Thus, for $x \in [0, 1] H(X, Y) = H((1 - x, x),(1 - y, y)) = (1 - x)(1 - y)a_{11} + (1 - x)ya_{12} + x(1 - y)a_{21} + +xya_{22} = x(-a_{11} + a_{11}y + a_{21} - a_{12}y - a_{21}y - a_{22}y) + (a_{11} - a_{11}y + a_{21}y)$

Given that y is fixed, the function Hb(x) is a straight line, Hb(x) = ax + b, so obviously the maximum and minimum of the function will be attained at the borders (either where x = 0 or where x = 1). The same result clearly holds if x is fixed and we let y vary.

Max-Min Strategy

Let Nash have a strategy $X = (1 - x, x)$, where x is the probability he will put the bomb in the support plane. As it has just been seen, Neumann's best reply will either be the strategy (1, 0) or (0, 1), so let's focus only on these cases. We will therefore have two possible payoffs when Nash uses X.

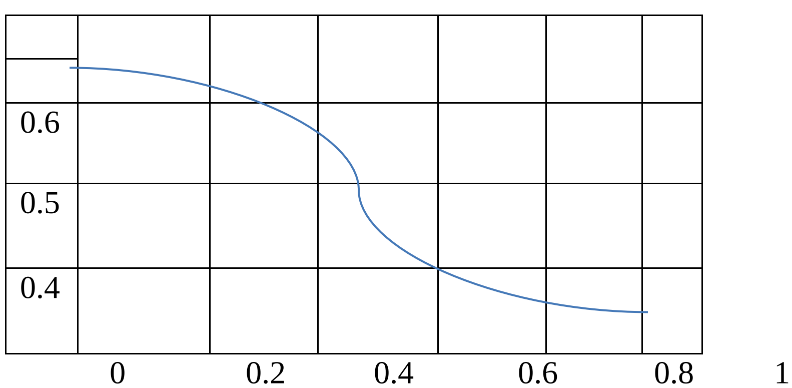
$r_1(x) = H(X,(1, 0)) = (0.7 - 0.6)x + 0.6 = 0.1x + 0.6$

$r_2(x) = H(X,(0, 1)) = -0.65x + 1$

Neumann, in order to protect his citizens, clearly wants to minimize Nash's gain, and therefore between these two possible options, he will always prefer the smaller one, so in the end the payoff will be:

$$H(X, Y) = H(x) = \min\{r_1(x), r_2(x)\}$$

we can see the graph of H(x). Since the intersection of $r_1(x)$ and r2(x) ($x* = 0, 533$ and $Hb(x*) = 0, 653$) is the higher point of the graph H(X, Y), it is Nash's wisest choice.



Graph of min{H(X,(1, 0)), H(X,(0, 1))}

Nash will maximize the function $\min\{r_1(x), r_2(x)\}$, thus he will maximize Neumann's minimal return, and that's why we call it the max-min strategy. Nash then will employ the strategy $X^* = (0.47, 0.53)$ and succeed, at least, 65.3% of the times.

Nevertheless, if he does not adhere to these recommendations, it is clear that

for $x \leq 0.533$ (when the bomb is in the bombarder more than 53.3% of the time) Neumann should always attack the bomborder

for $x \geq 0.533$ (when the bomb is in support plane more than 46.7% of the times) Neumann should attack the support plane.

Solution of the game

At the beginning, we said that Nash could guarantee an attack efficiency of 60% and Neumann could make sure the attack success rate didn't exceed 70%. Their guarantees were different. However, if we allow mixed strategies, the guarantees do coincide! This is a central theorem in Game Theory that we will prove in this thesis. When Nash employs his min-max strategy and Neumann his max-min strategy 2 we will see a 65.3% success rate of Nash's attacks. This will be called the value of the game. As we will see, this game is solved as we can give:

min-max strategy- $X^* = (0.47, 0.53)$

max-min strategy- $Y^* = (0.87, 0.13)$ value of the game $v = H(X^*, Y^*) = 0.653$

Conclusion

The reader should now wonder: doesn't this contradict Theorem 1.1? It was stated and proven that for any fixed strategy that your opponent picked, the optimal counterstrategy was a pure strategy. When player 1 picks X^*which is a fixed strategy, why do we suggest player 2 pick Y^*, which is not a pure strategy , The answer is that itcanbe readily checked that

$Hb(X, Y^*) = v \ \forall X \in Sb_1$

$Hb(X^*, Y) = v \ \forall Y \in Sb_2$

and given this remarkable property, it is convenient for every player to pick a max-min strategy because they guarantee certain results that other strategies fail to assure.

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