Average detour $D$-eccentricities of graphs

Venkateswara Rao, V.\textsuperscript{1}, Varma, P. L. N.\textsuperscript{1}, Reddy Babu. D\textsuperscript{2}

\textsuperscript{1} Division of Mathematics, Department of S&H, Vignan’s Foundation for Science, Technology and Research, Vadlamudi - 522 207, Guntur, India, E-mail:vunnamvenky@gmail.com, plnvarma@gmail.com

\textsuperscript{2} Department of Mathematics GST, GITAM University, Doddaballapura-561203, Bengaluru North, Karnataka, India, E-mail: reddybabu17@gmail.com

Abstract

The average distance in a graph is one of the important index which can be used in many applications. Eccentricity is the maximum distance from a vertex to any other vertex of the graph. The average eccentricity of a graph is applicable in network theory etc. In this article, we study average eccentricity of graph using Detour D-distance. We derive some properties and compute it for some classes of graphs.

Key words: Detour D-distance, detour D-eccentricity, average detour D-eccentricity.

1. Introduction

In this paper we discuss a topological index, namely, average eccentricity of vertices w.r.t. detour $D$-distance. The concept of $D$-distance in graphs was introduced by Reddy Babu and Varma in [4]. The concept of detour $D$-distance in graphs was introduced by V. Venkateswara Rao and Varma in [5]. Gupta. S, et. al. [1] and Ghorbanifar M, [2] explained connective eccentric index. The authors introduced $D$-eccentric and detour $D$-eccentric connectivity index (see [6, 7]).

In this article, we calculate the average eccentricities of vertices using detour $D$-distance in some families of the graphs.

Throughout this paper, all the graphs we consider are assumed to be finite, simple and connected. For any unexplained notation, see [3]. We recall some definitions based on detour $D$-distance, see [5].
The *detour D-distance*, \( D^D(u,v) \), between two vertices \( u, v \) of a connected graph \( G \) is defined as \( D^D(u,v) = \max \{ l^D(u,v) \} \) if \( u \) and \( v \) are distinct and \( D^D(u,v) = 0 \) if \( u = v \), where the maximum is taken over all \( u-v \) paths \( s \) in \( G \).

In a natural way, the *detour D-eccentricity* \( e^D_D(v) \) of \( v \) is the detour \( D \)-distance to a farthest vertex from \( v \). The *detour D-radius* \( r^D_D(G) \), *detour D-diameter* \( d^D_diam(G) \), are defined as the minimum and maximum eccentricities, respectively. The *detour D-center* \( C^D_D(G) \), and *detour D-periphery* \( P^D_D(G) \), of graph \( G \) consists of the set of vertices of minimum and maximum eccentricity, respectively. A graph \( G \) is *detour D-self centered* if \( V(G) = C^D_D(G) \).

The *average detour D-distance* between vertices is given by 
\[
\mu^D_D(G) = \frac{1}{n(n-1)} \sum_{u \neq v} D^D(u,v).
\]
The *detour D-distance matrix* of \( G \), denoted by \( M^D_D(G) \), is defined as 
\[
M^D_D(G) = [a_{i,j}]_{n \times n}
\]
where \( a_{i,j} = D^D(u_i,u_j) \) is the detour \( D \)-distance between the vertices \( u_i \) and \( u_j \). For any subset \( S \) of \( V(G) \), we define detour \( D \)-distance between a vertex \( u \) and \( S \) as 
\[
D^D(u,S) = \max \{ D^D(u,v) | v \in S \}.
\]
Further, we have \( \sigma^D_D(S) = \sum_{v \in V} D^D(v,S) \).

### 2. Results on average detour D-eccentricity

In a graph \( G \), the *average detour D-eccentricity* of \( G \) is defined as 
\[
avec^D_D(G) = \frac{1}{|G|} \sum_{v \in V} e^D_D(v).
\]

We begin with a result on detour D-self-centered graphs which is obvious.

**Theorem 2.1** For any graph, \( G \), 
\[
D^D - radius \leq \avec^D_D(G) \leq D^D - diameter.
\]

**Proof:** From the definition, minimum eccentricity is detour D-radius and maximum eccentricity is detour D-diameter. Hence 
\[
D^D - radius \leq \avec^D_D(G) \leq D^D - diameter.
\]

**Theorem 2.2** For detour D-self centered graph, \( G \), 
\[
D^D - radius = \avec^D_D(G) = D^D - diameter.
\]

**Proof:** From the definition of detour D-self centered graph, detour D-radius and detour D-diameter are same. Hence 
\[
D^D - radius = \avec^D_D(G) = D^D - diameter.
\]
Theorem 2.3  For any graph, $G$, $\text{avec}^D_D (G) \geq \mu^D_D (G)$.

Proof: Let $G$ be a graph with $n$ vertices. Then the detour $D$-distance matrix will have $n$ rows. The average detour $D$-distance of each row is less than or equal to the detour $D$-eccentricity of the row, i.e., $\frac{1}{n-1} \sum_{j=1}^{n} a_{i,j} \leq e^D_D (v_i)$ for each $i$. Here, $a_{i,j}$ stands for the detour $D$-distance between the vertices $v_i$ and $v_j$. Then taking the sum over all $i$, we get $\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \leq \sum_{i=1}^{n} e^D_D (v_i)$. Then

$$\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \leq \frac{1}{n} \sum_{i=1}^{n} e^D_D (v_i).$$

Hence $\mu^D_D (G) \leq \text{avec}^D_D (G)$.

Theorem 2.4  Let $G$ be a graph with $n$ vertices. Then $\text{avec}^D_D (G) \leq r^D_D (G) + \frac{1}{n} \left[ \sigma^D_D \left( C^D_D (G) \right) \right]$.  

Proof: Let $u_1, u_2, \ldots, u_n$ be the vertices of $G$. Some of the $u_i - u_j$ detour paths will pass through the detour $D$-center and some may not.

Let $P$ be a $u_i - u_j$ path in $G$ passing through a detour $D$-central vertex $u_k$ such that $D^D_D (u_i, u_k)$ is equal to detour $D$-radius, i.e., $D^D_D (u_i, u_k) = r^D_D (G)$. Also $C^D_D (G)$ is the set which contains all the central vertices. Clearly from the definition, detour $D$-distance from a vertex to $C^D_D (G)$ is always greater than detour $D$-distance from that vertex to the vertex $u_j$, i.e.,

$$D^D_D (C^D_D (G), u_j) \geq D^D_D (u_i, u_j).$$

Using triangle inequality we have $D^D_D (u_i, u_j) \leq D^D_D (u_i, u_k) + D^D_D (u_k, u_j) \leq r^D_D (G) + D^D_D (C^D_D (G), u_j)$.

Suppose $u_1^*, u_2^*, \ldots, u_n^*$ denote the detour $D$-eccentricity vertices of $u_i, u_2, \ldots, u_n$ respectively. Then

$$e^D_D (u_i) \leq r^D_D (G) + D^D_D \left( C^D_D (G), u_i^* \right).$$

Further, $\text{avec}^D_D (G) = \frac{1}{n} \left[ e^D_D \left( u_1 \right) + e^D_D \left( u_2 \right) + \ldots + e^D_D \left( u_n \right) \right]$ 

$$\leq \frac{1}{n} \left[ r^D_D (G) + D^D \left( C^D_D (G), u_1^* \right) + r^D_D (G) + D^D \left( C^D_D (G), u_2^* \right) + \ldots + r^D_D (G) + D^D \left( C^D_D (G), u_n^* \right) \right]$$

$$\leq \frac{1}{n} \left[ n r^D_D (G) + \sum_{i=1}^{n} D^D \left( C^D_D (G), u_i^* \right) \right] \leq r^D_D (G) + \frac{1}{n} \sum_{i=1}^{n} D^D \left( C^D_D (G), u_i^* \right).$$

Hence $\text{avec}^D_D (G) \leq r^D_D (G) + \frac{1}{n} \left[ \sigma^D_D \left( C^D_D (G) \right) \right]$.  

3
Theorem 2.5 Let $H$ be a spanning subgraph of a graph $G$ we have $\text{av}e^D_G(G) \geq \text{av}e^D_H(H)$.

Proof: If $H$ be the spanning subgraph of the graph $G$ then $H \subseteq G$ with same number of vertices and $E(H) \subseteq E(G)$. Clearly, as the number of edges may reduce in $H$, we have $\deg_G(v) \geq \deg_H(v)$. Hence $\text{av}e^D_G(G) \geq \text{av}e^D_H(H)$.

3. Average detour $D$-eccentricity of some families of graphs

Next, in this section, we calculate the average detour $D$-eccentricity of some classes of graphs. For these computations, we use the detour $D$-eccentricities of vertices of some graphs, which can be found in [5].

Theorem 3.1 The average detour $D$-eccentricity of the complete graph, $K_n$, is $n^2 - 1$.

Proof: In a complete graph $K_n$, each vertex is of degree $n - 1$. Thus eccentricity of each vertex is $n^2 - 1$. Hence the average detour D-eccentricity $\text{av}e^D(K_n) = \frac{1}{|K_n|} \sum_{v_i \in V} e^D(v_i) = \frac{1}{n} \sum_{i=1}^{n} e^D(v_i)$

$$= \frac{1}{n} (n e^D(v_i)) = e^D(v_i) = n^2 - 1.$$  

Theorem 3.2 For the cycle graph $C_n$, the average detour $D$-eccentricity is $3n - 1$.

Proof: In the cycle graph $C_n$ with $n$ vertices, we have detour $D$-eccentricity of is $3n - 1$. Then the average detour D-eccentricity is $\text{av}e^D(C_n) = \frac{1}{|C_n|} \sum_{v_i \in V} e^D(v_i) = \frac{1}{n} \sum_{i=1}^{n} e^D(v_i) = \frac{1}{n} (n e^D(v_i)) = e^D(v_i)$. Thus $\text{av}e^D(C_n) = 3n - 1$.

Theorem 3.3 The average detour $D$-eccentricity of the wheel graph $W_{1,n}$ with $n + 1$ vertices is $5n$.

Proof: In the wheel graph $W_{1,n}$ with $n + 1$ vertices let $v_0$ be the vertex which is adjacent to all other vertices. Then $\deg(v_0) = n$ and detour D-eccentricity is $5n$. All the remaining $n$ vertices have degree 3 and $D$-eccentricity is $5n$. Thus the average detour D-eccentricity of the wheel graph $W_{1,n}$ is $\text{av}e^D(W_{1,n}) = \frac{1}{|W_{1,n}|} \sum_{v_i \in V} e^D(v_i) = \frac{1}{n+1} \left[ \sum_{i=1}^{n} e^D(v_i) + 5n \right] = \frac{1}{n+1} \left[ \sum_{i=1}^{n} 5n + 5n \right] = 5n.$
Theorem 3.4 Let $K_{m,n} (m < n)$ be the complete bipartite graph with $m+n$ vertices. Then the average detour $D$-eccentricity is $$\frac{m(m+n)^2 + m^2(n+3) + n(2m-1)}{m+n}.$$ 

Proof: In the complete bipartite graph $K_{m,n} (m < n)$ with $m+n$ vertices, the vertex set $V$ can be partitioned as $V_1 \cup V_2$ with $V_1$ contains $m$ vertices and $V_2$ contains $n$ vertices. The detour $D$-eccentricity of all vertices in $V_1$ is $m^2 + mn + 3m$ and detour $D$-eccentricity of all vertices in $V_2$ is $m^2 + mn + 2m - 1$. Then the average detour $D$-eccentricity is $\text{avec}^D(K_{m,n}) = \frac{1}{|K_{m,n}|} \sum_{v \in V} e^D(v) = \frac{1}{m+n} \left[ (m(m^2 + mn + 3m) + n(m^2 + mn + 2m - 1)) \right] = \frac{m(m+n)^2 + m^2(n+3) + n(2m-1)}{m+n}.$$

Theorem 3.5 For the complete bipartite graph, $K_{m,m}$, with $2m$ vertices, the average detour $D$-eccentricity is $2m^2 + 2m - 1$.

Proof: In the complete bipartite graph $K_{m,m}$ with $2m$ vertices, the vertex set $V$ can be partitioned as $V_1 \cup V_2$ with $V_1$ contains $m$ vertices and $V_2$ contains $m$ vertices. Then in $K_{m,m}$ degree of each vertex is $m$ and $D$-eccentricity of each vertex $2m^2 + 2m - 1$. Thus the average detour $D$-eccentricity is $\text{avec}^D(K_{m,m}) = \frac{1}{|K_{m,m}|} \sum_{v \in V} e^D(v) = \frac{1}{2m} \left( \sum_{i=1}^{2m} (2m^2 + 2m - 1) \right) = 2m^2 + 2m - 1.$$

Theorem 3.6 For the Path graph, $P_n$, the average detour $D$-eccentricity of is $\frac{1}{2n} \left( 3n^2 + 8n - 16 \right)$ if $n$ is even and $\frac{1}{2n} \left( 3n^2 + 5n - 10 \right)$ if $n$ is odd.

Proof: In the path graph $P_n$ with $n$ vertices, the two end vertices have degree 1 and detour $D$-eccentricity $3(n-1)$. The remaining $(n-2)$ vertices have degree 2 and detour $D$-eccentricity
\[
\frac{3n-1}{2} \text{ if } n \text{ is odd and } \frac{3n+2}{2} \text{ if } n \text{ is even. We calculate average detour } D\text{-eccentricity by consider the even and odd cases separately.}
\]

Case (i) \( n \) is odd

The average detour \( D\)-eccentricity is

\[
avec_D^P (P_n) = \frac{1}{|P_n|} \sum_{v \in V} e_D^P (v_i) = \frac{1}{n+1} \left[ 3(n-1) + \sum_{i=2}^{n} \frac{3n-1}{2} + 3(n-1) \right] = \frac{1}{2n} \left( 3n^2 + 5n - 10 \right).
\]

Case (ii) \( n \) is even

The average detour \( D\)-eccentricity is

\[
avec_D^P (P_n) = \frac{1}{|P_n|} \sum_{v \in V} e_D^P (v_i) = \frac{1}{n+1} \left[ 3(n-1) + \sum_{i=2}^{n} \frac{3n+2}{2} + 3(n-1) \right] = \frac{1}{2n} \left( 3n^2 + 8n - 16 \right).
\]

**Theorem 3.7** For the Star graph, \( St_{1,n} \), with \( n+1 \) vertices the average detour \( D\)-eccentricity is

\[
\frac{n^2 + 5n + 2}{n+1}.
\]

Proof: In the star graph \( St_{1,n} \), the degree of central vertex is \( n \) and detour \( D\)-eccentricity is \( n+2 \). All the remaining \( n \) vertices have degree 1 and detour \( D\)-eccentricity \( n+4 \). Thus the average detour \( D\)-eccentricity is

\[
avec_D^P \left( St_{1,n} \right) = \frac{1}{|St_{1,n}|} \sum_{v \in V} e_D^P (v_i) = \frac{1}{n+1} \sum_{i=1}^{n+1} e_D^P (v_i) = \frac{1}{n+1} \left( (n+2) + n(n+4) \right) = \frac{n^2 + 5n + 2}{n+1}.
\]

**References**
