Estimation of Stress-Strength Reliability for Inverse Rayleigh Distribution and Power Function Distribution by Transformation Method

Surinder Kumar and Ankit

Department of Statistics

School of Physical and Decision Sciences

Babasaheb Bhimrao Ambedkar University

Lucknow-226025. India.

e-mail: ankit0209@gmail.com

Abstract: In this article, the problem of estimating the reliability R = Pr (X < Y) in the situation, when X and Y are the independent random variables from inverse Rayleigh distribution, is considered. R = Pr (X < Y) represents the Stress and Strength model which arises in the case when one wishes to measure the performance of a mechanical system. Deviating from the conventional theories of estimation, the transformation methods are used to derive the MLE, UMVUE and Confidence interval for the model under consideration. Also the situation, when Stress follows an inverse Rayleigh distribution and Strength follows a Power function distribution, is considered and the problem of Stress-Strength reliability model is further studied through establishing the relationship among the parameters of the distributions involved in the model.

Keywords: Inverse Rayleigh distribution, Power function distribution, Stress-Strength model, Transformation method, MLE, UMVUE and Confidence Interval.

1. Introduction

The inverse Rayleigh distribution is an important lifetime distribution in reliability and survival analysis. It has many applications in the area of reliability studies. The probability density function of the inverse Rayleigh distribution with scale parameter λ is defined as

$$f(x\,|\,\lambda) = \frac{2\lambda}{x^3} exp\left\{-\frac{\lambda}{x^2}\right\};\ x, \lambda > 0 \tag{1.1}$$

where λ is assumed to be unknown constant.

The corresponding cumulative distribution function is,

$$F(x|\lambda) = exp\left\{-\frac{\lambda}{x^2}\right\};\ x, \lambda > 0 \tag{1.2}$$

Initially, this distribution was introduced by Trayer (1964) and discussed its application in reliability theory. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution and also discussed some properties of the maximum likelihood estimator for the unknown parameter. Gharraph (1993) derived the measures of location (Mean, Harmonic mean, Geometric mean, Mode, and the Median) for the inverse Rayleigh distribution and estimated its scale parameter by using various methods of estimation and compared the results numerically in term of their bias and MSE. Mukherjee and Saran (1984) studied the hazard rate of inverse Rayleigh distribution and found that it was increasing when x < 1.069543*λ, decreasing when x > 1.069543*λ and tended to become stable as x increased. Mukherjee and Maiti (1996) studied some properties of the inverse Rayleigh distribution and developed various methods of estimation for the inverse Rayleigh distribution. Abd-el-Monem (2003) developed some estimation and prediction results for the inverse Rayleigh distribution. A comparison of the negative moment estimator with maximum likelihood estimator of the inverse Rayleigh distribution was studied by Mohsin and Shahbaz (2005). Rao *et al.* (2013) considered the problem of estimating the Stress-Strength reliability R = Pr (X < Y) in the inverse Rayleigh distribution for complete sample.

In this paper, deviating from the conventional theories of estimation, the transformation methods are used to derive the MLE, UMVUE and Confidence interval for the inverse Rayleigh distribution to estimate the reliability function R = Pr (X < Y) in the situation, when X and Y are the independent random variables (r.v.'s) from inverse Rayleigh distribution.

A lot of work has been done by various authors on the measure R = Pr (X < Y), for a brief review one may refers to Tong (1974), Sathe, S. P. Shah (1981), Awad and Gharraf (1986) and Constantine, Karson and Tse (1986).

2. Estimation of $R = Pr(Y < X)$ for Inverse Rayleigh Distribution

Theorem 2.1: The MLE of reliability function $R = Pr(Y < X)$ is given by

$$\tilde{R}_{YX} = \frac{T_y}{T_x + T_y} \tag{2.1}$$

where,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{{x_i}^2}$$

and

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{{y_i}^2}$$

Proof: Let us consider the transformation $\epsilon = \frac{1}{x^2}$ in equation (1.1), we get

$$f(\epsilon|\lambda) = \lambda\, exp\{-\lambda\epsilon\};\ \epsilon, \lambda > 0 \tag{2.2}$$

which is the pdf of exponential distribution with mean $1/\lambda$.

Let us consider X and Y two independent r.v.'s following inverse Rayleigh distribution, given in eq. (1.1), with parameters λ_1 and λ_2, respectively. If we consider the transformations $\epsilon = \frac{1}{x^2}$ and $\eta = \frac{1}{y^2}$, then we have two independent exponential variates ϵ and η with means $1/\lambda_1$ and $1/\lambda_2$, respectively and the corresponding reliability function $R_{YX} = Pr\,(Y < X)$, becomes $R_{\epsilon\eta} = Pr(\epsilon < \eta)$.

Recall, that the MLE of $R_{\epsilon\eta}$ based on samples $\underline{\epsilon}$ and $\underline{\eta}$ of size n_1 and n_2, respectively, will be given in the form

$$\tilde{R}_{\epsilon\eta} = \frac{\bar{\eta}}{\bar{\epsilon} + \bar{\eta}} \tag{2.3}$$

where,

$$\bar{\epsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1}\epsilon_i$$

and

$$\bar{\eta} = \frac{1}{n_2}\sum_{i=1}^{n_2}\eta_i$$

replacing ϵ_i by $\frac{1}{{x_i}^2}$ and η_i by $\frac{1}{{y_i}^2}$ in the eq. (2.3), we get

$$\tilde{R}_{\epsilon\eta} = \frac{T_y}{T_x + T_y} = \tilde{R}_{YX} \tag{2.4}$$

where, T_x and T_y are,

$$T_x = \frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{{x_i}^2} = \frac{1}{n_1}\sum_{i=1}^{n_1}\epsilon_i = \bar{\epsilon},$$

$$T_y = \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{{y_i}^2} = \frac{1}{n_2}\sum_{i=1}^{n_2}\eta_i = \bar{\eta}$$

Theorem 2.2: The UMVUE of reliability function $R = p\,(Y < X)$ is given by

$$\hat{R}_{YX} = \begin{cases} Q_1\left(n_1, n_2, n_1T_x, n_2T_y\right); & if\ n_1T_x \geq n_2T_y \\ Q_2\left(n_1, n_2, n_1T_x, n_2T_y\right); & if\ n_1T_x < n_2T_y \end{cases} \tag{2.5}$$

where, Q_1 and Q_2 are defined as

$$Q_1(a, b, u, v) = \sum_{i=0}^{a-2} (-1)^i \frac{\Gamma a \Gamma b}{\Gamma (a - i - 1) \Gamma (b + i + 1)} \left(\frac{v}{u}\right)^{i+1}$$

$$Q_2(a, b, u, v) = \sum_{i=0}^{b-1} \frac{\Gamma a \Gamma b}{\Gamma (a + i) \Gamma (b - i)} \left(\frac{u}{v}\right)^{i}$$

and T_x and T_y are

$$T_x = \frac{1}{n_1} \sum_{i=1}^{n_1} \frac{1}{{x_i}^2}, \qquad T_y = \frac{1}{n_2} \sum_{i=1}^{n_2} \frac{1}{{y_i}^2}$$

Proof: To obtain the UMVUE of reliability function $R = Pr\ (Y < X)$ based on samples $\underline{X}$ and $\underline{Y}$ of size n_1 and n_2, respectively, where X and Y are two independent r.v.'s following inverse Rayleigh distribution given in equation (1.1) with parameters $\lambda_1$and λ_2, respectively. We consider the transformations $\epsilon = \frac{1}{x^2}$ and $\eta = \frac{1}{y^2}$. Now, we have two independent exponential variates ϵ and η with means $1/\lambda_1$ and $1/\lambda_2$, respectively and the corresponding reliability function R_{YX}, becomes $R_{\epsilon\eta}$. Recall, that the UMVUE of reliability function $R_{\epsilon\eta} = Pr(\epsilon < \eta)$ will be given by

$$\hat{R}_{\epsilon\eta} = \begin{cases} Q_1(n_1, n_2, n_1\bar{\epsilon}, n_2\bar{\eta}); & if\ n_1\bar{\epsilon} \geq n_2\bar{\eta} \\ Q_2(n_1, n_2, n_1\bar{\epsilon}, n_2\bar{\eta}); & if\ n_1\bar{\epsilon} < n_2\bar{\eta} \end{cases} \tag{2.6}$$

where, Q_1 and Q_2 are defined as

$$Q_1(a, b, u, v) = \sum_{i=0}^{a-2} (-1)^i \frac{\Gamma a \Gamma b}{\Gamma (a - i - 1) \Gamma (b + i + 1)} \left(\frac{v}{u}\right)^{i+1}$$

$$Q_2(a, b, u, v) = \sum_{i=0}^{b-1} \frac{\Gamma a \Gamma b}{\Gamma (a + i) \Gamma (b - i)} \left(\frac{u}{v}\right)^{i}$$

here,

$$\bar{\epsilon} = \frac{1}{n_1} \sum_{i=1}^{n_1} \epsilon_i = \frac{1}{n_1} \sum_{i=1}^{n_1} \frac{1}{{x_i}^2} = T_x$$

and

$$\bar{\eta} = \frac{1}{n_2}\sum_{i=1}^{n_2} \eta_i = \frac{1}{n_2}\sum_{i=1}^{n_2} \frac{1}{{y_i}^2} = T_y$$

replacing ϵ_i and η_i by $\frac{1}{{x_i}^2}$ and $\frac{1}{{y_i}^2}$, respectively, in the eq. (2.6), we get

$$\hat{R}_{\epsilon\eta} = \hat{R}_{YX} = \begin{cases} Q_1(n_1, n_2, n_1T_x, n_2T_y); & if\ n_1T_x \geq n_2T_y \\ Q_2(n_1, n_2, n_1T_x, n_2T_y); & if\ n_1T_x < n_2T_y \end{cases}$$

Hence, the theorem follows.

Theorem 2.3: The asymptotic confidence interval for reliability function $R = Pr\ (Y < X)$ with confidence coefficient $(1-\gamma)$ is $(\hat{R}_{YX} - z_{\gamma/2}\hat{\sigma}_{YX}, \hat{R}_{YX} - z_{\gamma/2}\hat{\sigma}_{YX})$, where $z_{\gamma/2}$is the $(1-\frac{\gamma}{2})$ percentile of standard normal distribution and

$$\hat{\sigma}_{YX}{}^2 = \widehat{var}(\hat{R}_{YX}) = (\hat{R}_{YX})^2 - \frac{(n_1-1)(n_1-2)(n_2-1)(n_2-2)}{{n_1}^2{n_2}^2(T_x)^{n_1-1}(T_y)^{n_2-1}} H(n_1, n_2, T_x, T_y) \tag{2.7}$$

and $H(n_1, n_2, \bar{\epsilon}, \bar{\eta})$ is defined as

$$H(n_1, n_2, \bar{\epsilon}, \bar{\eta}) = \iint (\bar{\epsilon} - \frac{\epsilon_1 + \epsilon_2}{n_1})^{n_1-3}(\bar{\eta} - \frac{\eta_1 + \eta_2}{n_2})^{n_2-3}\, d\epsilon_1\, d\epsilon_2\, d\eta_1\, d\eta_2 \tag{2.8}$$

Integral is over the region W^* defined as

$$W^* = \{(\epsilon_1, \epsilon_2, \eta_1, \eta_2): \epsilon_1 + \epsilon_2 < n_1\bar{\epsilon}, \eta_1 + \eta_2 < n_2\bar{\eta}, 0 < \eta_1 < \epsilon_1, 0 < \eta_2 < \epsilon_2\}$$

Proof: To construct the confidence interval for reliability function $R = Pr\ (Y < X)$, we need to obtain the UMVUE for variance of $\hat{R}_{YX}$. Here, X and Y are independent r.v.'s following inverse Rayleigh distribution with parameters $\lambda_1$and λ_2, respectively. Consider the transformations $\epsilon = \frac{1}{x^2}$ and $\eta = \frac{1}{y^2}$. Then ϵ and η will come out independent exponential variates with means $1/\lambda_1$ and $1/\lambda_2$, respectively and the corresponding reliability function R_{YX}, becomes $R_{\epsilon\eta}$. We know that for exponential variates ϵ and η, the UMVUE of variance of $\hat{R}_{\epsilon\eta}$, based on samples $\underline{\epsilon}$ and $\underline{\eta}$ of size n_1 and n_2, respectively, is given as

$$\hat{\sigma}_{\epsilon\eta}{}^2 = \widehat{var}(\hat{R}_{\epsilon\eta}) = (\hat{R}_{\epsilon\eta})^2 - \frac{(n_1-1)(n_1-2)(n_2-1)(n_2-2)}{{n_1}^2{n_2}^2(\bar{\epsilon})^{n_1-1}(\bar{\eta})^{n_2-1}} H(n_1, n_2, \bar{\epsilon}, \bar{\eta}) \tag{2.9}$$

when $H(n_1, n_2, \bar{\epsilon}, \bar{\eta})$ is defined as

$$H(n_1, n_2, \bar{\epsilon}, \bar{\eta}) = \iint (\bar{\epsilon} - \frac{\epsilon_1 + \epsilon_2}{n_1})^{n_1-3} (\bar{\eta} - \frac{\eta_1 + \eta_2}{n_2})^{n_2-3} \, d\epsilon_1 \, d\epsilon_2 \, d\eta_1 \, d\eta_2 \tag{2.10}$$

Integral is over the region W^* defined as

$$W^* = \{(\epsilon_1, \epsilon_2, \eta_1, \eta_2): \epsilon_1 + \epsilon_2 < n_1\bar{\epsilon}, \eta_1 + \eta_2 < n_2\bar{\eta}, 0 < \epsilon_1 < \eta_1, 0 < \epsilon_2 < \eta_2\}$$

where

$$\bar{\epsilon} = \frac{1}{n_1} \sum_{i=1}^{n_1} \epsilon_i$$

and

$$\bar{\eta} = \frac{1}{n_2} \sum_{i=1}^{n_2} \eta_i$$

replacing ϵ_i by $\frac{1}{{x_i}^2}$ and η_i by $\frac{1}{{y_i}^2}$ in the eq. (2.9), we will get the UMVUE for variance of $\hat{R}_{YX}$, i.e., ${\hat{\sigma}_{YX}}^2$ as follows

$${\hat{\sigma}_{\epsilon\eta}}^2 = {\hat{\sigma}_{YX}}^2 = \widehat{var}(\hat{R}_{YX}) = (\hat{R}_{YX})^2 - \frac{(n_1 - 1)(n_1 - 2)(n_2 - 1)(n_2 - 2)}{{n_1}^2 {n_2}^2 (T_x)^{n_1-1} (T_y)^{n_2-1}} H(n_1, n_2, T_x, T_y)$$

The asymptotic confidence interval for reliability function $R_{YX} = Pr\,(Y < X)$ with confidence coefficient $(1 - \gamma)$is $(\hat{R}_{YX} - z_{\gamma/2}\hat{\sigma}_{YX}, \hat{R}_{YX} - z_{\gamma/2}\hat{\sigma}_{YX})$, where $\hat{\sigma}_{YX}$ is defined above and $z_{\gamma/2}$ is the $(1 - \frac{\gamma}{2})$ percentile of standard normal distribution.

Hence, the theorem follows.

3. Power Function Distribution

The problem of Stress-Strength testing i.e. $R = Pr(Y > X)$, is considered for the situation when the Stress follows inverse Rayleigh distribution and the Strength follows Power function distribution.

Let us assume that the r.v. X represents the Stress that an item faces, follows inverse Rayleigh distribution and Strength Y follows Power function distribution with pdf given by

$$g(y) = \frac{\mu}{\theta}\left(\frac{y}{\theta}\right)^{\mu-1}; 0 < y < \theta, \mu > 0, \tag{3.1}$$

where θ and μ are scale and shape parameters respectively.

3.1. Strength Reliability for Finite Strength

It is justifiable to take strength as a finite probability distribution i.e. Power function distribution, as Strength of items/devices may always limits to a finite range. Thus, the maximum possible value of Strength distribution is θ. Y cannot exceed X, if X exceeds θ. The total unreliability of the items is therefore, obtained by $Pr(X > \theta)$. Alam and Roohi (2003) have termed this as probability of "disaster".

Theorem 3.1: If the random variable X and Y follows the inverse Rayleigh distribution and Power function distribution respectively, then $\alpha = Pr(X > \theta)$ is given by

$$\alpha = Pr(X > \theta)$$
$$= 1 - e^{-\lambda/\theta^2} \quad (3.2)$$

Proof: We know that

$$\alpha = Pr(X > \theta)$$
$$= \int_{\theta}^{\infty} \frac{2\lambda}{x^3} exp\left\{-\frac{\lambda}{x^2}\right\} dx \quad (3.3)$$

on substituting $t = \lambda/x^2$, in eq. (3.3), we get

$$\alpha = \int_0^{\lambda/\theta^2} e^{-t} dt$$

or,

$$\alpha = 1 - e^{-m}$$

where, $m = \lambda/\theta^2$.

Hence, the theorem follows.

3.2. Numerical study for the Probability of Disaster for different values of $'\lambda'$ and $'\theta'$

A numerical study for the probability of disaster $\alpha = Pr(X > \theta)$ for different values of λ and θ is done. The computed values are presented in Table 1 and it is interpreted from Table 1 that probability of disaster increase with an increase in values of λ and a decreases with an increase in the values of θ.

Table 1: Numerical values for $\alpha = Pr(X > \theta)$ for different combinations of $'\lambda'$ and $'\theta'$

λ\θ	**0.50**	**1.50**	**2.50**	**5.0**	**7.50**	**10.0**	**12.50**	**15.0**	**17.50**	**20.0**
0.05	0.181	0.022	0.008	0.002	0.0009	0.0005	0.00032	0.00022	0.00016	0.00012

0.10	0.330	0.045	0.016	0.004	0.002	0.001	0.0006	0.00044	0.0003	0.00025
0.15	0.451	0.065	0.023	0.006	0.003	0.0015	0.001	0.00067	0.0005	0.00038
0.20	0.551	0.085	0.315	0.008	0.004	0.002	0.001	0.0009	0.0007	0.0005
0.25	0.632	0.105	0.039	0.010	0.004	0.0025	0.002	0.001	0.0008	0.0006
0.5	0.865	0.199	0.077	0.020	0.009	0.005	0.003	0.002	0.0016	0.0013
0.75	0.950	0.284	0.113	0.030	0.013	0.007	0.005	0.003	0.0025	0.0019
1.0	0.982	0.359	0.148	0.039	0.018	0.001	0.006	0.004	0.0033	0.0025
1.25	0.993	0.426	0.181	0.049	0.022	0.012	0.008	0.006	0.0041	0.0031
1.5	0.998	0.487	0.671	0.058	0.026	0.015	0.010	0.007	0.0049	0.0037
1.75	0.999	0.541	0.244	0.068	0.031	0.017	0.011	0.008	0.0056	0.0044
2.0	0.9997	0.589	0.274	0.077	0.035	0.0198	0.127	0.009	0.0065	0.0050

Alternatively, we may also obtain the numerical values of m $(=\lambda/\theta^2)$ for fixed α at different tolerance levels. Further, these values are used to obtain the optimum cost for manufacturing of items at desired tolerance level.

Table 2: Values of ‘m’ at different levels of ‘α’

α	0.9	0.7	0.5	0.25	0.1	0.05	0.01	0.001	0.0001
$m=\lambda/\theta^2$	2.303	1.204	0.693	0.288	0.105	0.051	0.010	0.001	0.0000

Remarks:

1. It is concluded from Table 1 that the values of the probability of disaster decrease as θ increase for fixed λ and increase as λ increase for fixed θ.

2. Table 2 shows that the probability of disaster increase or decrease with the value of ratio of λ and θ increase or decrease. In other words, probability of disaster decrease as θ increase.

3.3. Stress and Strength Reliability when the r.v.’s X and Y follows Inverse Rayleigh distribution and Power function distribution respectively:

For the Stress-Strength model the probability, $P=Pr(Y>X)$, when the random variable X and Y follows inverse Rayleigh distribution and Power function distribution respectively, is given by the following theorem.

Theorem 3.2: $P=Pr(Y>X)$ is given by

$$P = e^{-\lambda/\theta^2} - \frac{\lambda^{\mu/2}}{\theta^{\mu}} \int_{\lambda/\theta^2}^{\infty} t^{-\frac{\mu}{2}} e^{-t} dt$$

or,

$$P = e^{-m} - m^{\mu/2} \int_{m}^{\infty} t^{\left(1-\frac{\mu}{2}\right)-1} e^{-t} dt \tag{3.4}$$

where, $m = \frac{\lambda}{\theta^2}$.

Proof:

$$Pr(Y > X) = \int_{x=0}^{\theta} \int_{y=x}^{\theta} f(x;\lambda) g(y;\mu,\theta) dy dx$$

$$= \int_{x=0}^{\theta} \int_{y=x}^{\theta} \frac{2\lambda}{x^3} exp\left\{-\frac{\lambda}{x^2}\right\} \frac{\mu}{\theta} \left(\frac{y}{\theta}\right)^{\mu-1} dy dx$$

$$= \int_{0}^{\theta} \frac{2\lambda}{x^3} exp\left\{-\frac{\lambda}{x^2}\right\} \frac{1}{\theta^{\mu}} (\theta^{\mu} - x^{\mu}) dx$$

$$= \int_{0}^{\theta} \frac{2\lambda}{x^3} exp\left\{-\frac{\lambda}{x^2}\right\} dx - \int_{0}^{\theta} \frac{2\lambda}{x^3} \frac{x^{\mu}}{\theta^{\mu}} exp\left\{-\frac{\lambda}{x^2}\right\} dx \tag{3.5}$$

on substituting, $t = \lambda/x^2$ in (3.5), we get

$$Pr(Y > X) = \int_{\infty}^{\lambda/\theta^2} -e^{-t}\, dt - \int_{\infty}^{\lambda/\theta^2} -\frac{\lambda^{\mu/2}}{\theta^{\mu}} t^{-\mu/2} e^{-t}\, dt$$

$$= e^{-\lambda/\theta^2} - \frac{\lambda^{\mu/2}}{\theta^{\mu}} \int_{\frac{\lambda}{\theta^2}}^{\infty} t^{-\frac{\mu}{2}} e^{-t} dt \tag{3.6}$$

putting, $m = \frac{\lambda}{\theta^2}$ in (3.6) we get

$$= e^{-m} - m^{\mu/2} \int_{m}^{\infty} t^{-\frac{\mu}{2}} e^{-t} dt$$

Hence, the theorem follows.

Table 3: Strength-reliability of an item for varying values of ‘m’ and ‘μ’

m\μ	1.8	1.9	2	2.25	2.5	2.75	3.0	3.5	4	5	10	20	100	400
0.0001	0.999	0.999	0.999	0.999	1	1	1	1	1	1	1	1	1	1
0.001	0.990	0.991	0.993	0.995	0.996	0.997	0.997	0.998	0.998	0.998	0.999	0.999	0.999	0.999
0.002	0.982	0.984	0.987	0.990	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.998	0.998	0.998
0.003	0.976	0.979	0.981	0.986	0.988	0.990	0.992	0.993	0.994	0.995	0.996	0.997	0.997	0.997

0.004	0.970	0.973	0.976	0.981	0.985	0.987	0.989	0.991	0.992	0.993	0.995	0.996	0.996	0.997
0.005	0.964	0.968	0.971	0.977	0.982	0.984	0.986	0.989	0.990	0.991	0.994	0.995	0.995	0.995
0.006	0.959	0.963	0.967	0.974	0.978	0.981	0.984	0.987	0.988	0.990	0.993	0.993	0.994	0.994
0.007	0.954	0.958	0.962	0.970	0.975	0.978	0.981	0.984	0.986	0.988	0.991	0.992	0.993	0.993
0.008	0.949	0.954	0.958	0.966	0.972	0.976	0.978	0.982	0.984	0.987	0.990	0.991	0.992	0.992
0.009	0.944	0.949	0.954	0.962	0.969	0.973	0.976	0.980	0.982	0.985	0.989	0.990	0.991	0.991
0.010	0.939	0.945	0.950	0.959	0.965	0.970	0.973	0.978	0.981	0.983	0.988	0.990	0.990	0.990
0.025	0.881	0.889	0.897	0.912	0.923	0.932	0.938	0.947	0.953	0.960	0.962	0.973	0.975	0.975
0.05	0.807	0.818	0.828	0.848	0.864	0.876	0.886	0.900	0.910	0.921	0.940	0.947	0.950	0.951
0.10	0.698	0.711	0.723	0.747	0.767	0.784	0.797	0.818	0.833	0.852	0.883	0.897	0.903	0.904
0.25	0.493	0.506	0.518	0.544	0.567	0.585	0.602	0.629	0.649	0.679	0.734	0.762	0.775	0.778
0.5	0.308	0.317	0.327	0.348	0.366	0.383	0.398	0.423	0.443	0.474	0.541	0.581	0.600	0.605
0.75	0.203	0.210	0.217	0.233	0.247	0.260	0.272	0.292	0.310	0.336	0.400	0.444	0.465	0.471
1.0	0.138	0.144	0.149	0.160	0.171	0.181	0.190	0.206	0.219	0.241	0.297	0.339	0.361	0.366
2.5	0.018	0.019	0.020	0.022	0.024	0.025	0.027	0.030	0.033	0.037	0.052	0.068	0.078	0.081
5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.005	0.006	0.007

Note: the expression (3.4) involved incomplete Gamma function whose numerical values are obtained by using *Mathematica Software*.

Discussion:

The Strength of manufactured components follows Power function distribution, which justifies the fact that Strength is always have an upper limit. Thus, it is likely that the possible values of θ may have an upper limit say θ_0. For example, the capacity of accelerating an engine must be subject to maximum possible speed. For a fixed tolerance level α, suppose θ_α is the desired value of θ. In case $\theta_\alpha < \theta_0$, we may obtain the required value of μ say μ_α, by using Table 3, so that the item is manufactured with the Strength distribution having parameters $(\mu_\alpha, \theta_\alpha)$ and consequently the desired Strength reliability is achieved. However, if $\theta_\alpha > \theta_0$, we will have to either adjust α or look for an alternate item.

An illustrative example

Let us assume that the minimum feasible value of m is 0.001. For $\alpha \leq 0.01$, we must have $m \geq 0.001$. Since m cannot be less than 0.001, we have the option of fixing the item in such a way that $0.001 \leq m \leq 0.01$ i.e. $\sqrt{100\lambda} \leq \theta \leq \sqrt{1000\lambda}$ or $10\sqrt{\lambda} \leq \theta \leq 31.62\sqrt{\lambda}$ (as $m = \lambda/\theta^2$ and $\theta > 0$) and corresponding value of μ leads to a maximum of $Pr(Y > X)$. The cost factor of adjusting the parameters may be taken into consideration, here as the cost of varying θ and μ may be different. Theoretically, the costs here may be increasing or decreasing function of θ and μ depending upon the nature of the parameters. Usually $C(Y)$ is an increasing function of Y if Y is the mean Strength. In our study $E(Y) = \frac{\mu\theta}{\mu+1}$, which implies that the mean Strength increases by increasing either of the two parameters. Hence, we may assume that both the costs are increasing functions of the respective parameters. Assuming the costs to be directly proportional to the required values of the parameters, the problem can be formulated as follows:

Let C_1 be the cost of adjusting one unit of θ and C_2 be the cost of adjusting one unit of μ. Minimize $C = C_1\theta + C_2\mu$ subject to condition $10\sqrt{\lambda} \le \theta \le 31.62\sqrt{\lambda}$ for given value of λ and $Pr(Y > X) \ge 0.99$.

The problem may be solved analytically replicating the method used in construction of Table 3 for $m = 0.001, 0.002, \dots, 0.01$ *i.e.* $\theta = 44{,}31, \dots, 14.1421$ and find those values of μ for which $Pr(Y > X) \ge 0.99$. Evaluate the cost function for each pair of (μ, θ):

Table 4: Table for obtaining the Optimum cost of manufacturing item

m	θ	μ	$C_1\theta + C_2\mu$
0.001	44	≥ 1.815	$44C_1 + 2C_2$
0.002	31	≥ 2.285	$31C_1 + 2.5C_2$
0.003	25	≥ 2.81	$25C_1 + 5C_2$
0.004	22	≥ 3.319	$22C_1 + 5C_2$
0.005	20	≥ 3.935	$20C_1 + 5C_2$
0.006	18	≥ 5.16	$18C_1 + 5C_2$
0.007	16	≥ 8.9695	$16C_1 + 10C_2$
0.008	15	≥ 17.31	$15C_1 + 10C_2$
0.009	14.9071	≥ 19.119	$14.90C_1 + 25C_2$
0.01	14.1421	≥ 395.767	$14.14C_1 + 2C_2$

It is concluded from the Table 4 that for θ = 44, μ ≥ 2; for θ = 31, μ ≥ 2.5; and so on, the required objective i.e. $Pr(Y > X) \ge 0.99$ has been achieved.

References

[1] Abd-el-Monem, A. A. (2003): Estimation and Prediction for the Inverse Rayliegh life distribution. M.Sc. Thesis. Faculty of Education, Ain Shames University.

[2] Alam and Roohi (2003): On facing an exponential stress with strength having Power function distribution. *Aligarh J. of Stat.*, **23**, 57-63.

[3] Awad, M. and Gharraf, M. K. (1986): Estimation of P(Y < X) in the burr case: A comparative study. *Commun. Statist. B-Simul. Comp.*, **15** (2), 389-403.

[4] Constantine, K., Karson, M. and Tse, S. K. (1986): Estimators of P(Y < X) in the Gamma case. *Commun. Statist. -Simul. Comp.*, **15**, 365-388.

[5] Gharraph, M. K. (1993): Comparison of Estimation of Location Measure of an Inverse Rayleigh Distribution. *The Egyptian Statistical journal*, **37**, 295-309.

[6] Mohsin and Shahbaz (2005): Comparison of Negative Moment Estimator with Maximum Likelihood Estimator of Inverse Rayleigh Distribution. *PJSOR*, **1**, 45-48.

[7] Mukherjee, S. and Saran, L. (1984): Bivariate inverse Rayleigh Distributions in reliability studies. *J. Ind. Statist. Assoc.*, **22**, 23-31.

[8] Mukherjee, S. P. and Maiti, S. S. (1996): A Percentile Estimator of the Inverse Rayleigh Parameter. *IAPQR Transactions*, **21**, 63-65.

[9] Rao, G. S., Kantam, R. R. L., Rosaiah, K. and Reddy, J. P. (2013): Estimation of stress-strength reliability from inverse Rayleigh distribution. *Journal of Industrial and Production Engineering*, **30** (4), 256–263.

[10] Sathe, Y. S. and Shah, S. P. (1981): On estimation P(Y < X) for the exponential distribution. *Commun. Statist. -Theor. Meth.*, **10**, 39-47.

[11] Tong, H. (1974): A note on the estimation of P(Y < X) in the exponential case. *Technometrics*, **16**, 625.

[12] Treyer, V. (1964): *Proceedings of the Academy of Sciences. Belerus*, U.S.S.R.

[13] Voda, V. (1972): On the Inverse Rayleigh Distributed Random Variable. *Rep. Stat. Appl. Res. JUSE*, **19** (4), 13-21.