DISCUSSION ON PRIME CORDIAL LABELING

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Abstract

This paper provides a special labeling on path P_k called $P_k[a, b, n]$. A discussion on prime cordial labeling of graphs obtained by identifying each end vertex of a path P_k, with corresponding vertices of each of the two copies of a graph G is done using the concept of odd prime labeling of graphs and the labeling $P_k[a, b, n]$.

Keywords: graph labeling, prime cordial graph, odd prime graph

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1 Introduction

In this paper only finite, simple, undirected, non-trivial and connected graphs are considered. The vertex and edge sets are respectively denoted by V or $V(G)$ and E or $E(G)$. Such graphs are denoted by $G(V, E)$. $|V|$ and $|E|$ denotes respectively the number of elements in V and E. Clark and Holton [8] is referred here for graph theoretical terminologies and notations. Burton [2] is referred for number theoretical notations. The paper discusses prime cordial labeling on the family of graphs which are obtained by identifying each end vertex of a path, with corresponding vertices of each of the two copies of an odd prime graph.

We start with a brief summary of definitions and notations required for this paper.

Notation 1.1. [12, 10] For any natural number n, $[n]$ and O_n denotes the set of first n natural numbers and first n odd natural numbers respectively. i.e. $[n] = \{1, 2, 3, \ldots, n\}$ and $O_n = \{1, 3, 5, \ldots, 2n - 1\}$.

Definition 1.2. [6] An assignment of integers to vertices and/or edges of a graph, subject to certain conditions is known as *graph labeling*.

In 1960, since the graph labeling has been introduced, more than 200 graph labeling techniques have been studied. For latest survey of graph labeling, Gallian [6] is referred.

Definition 1.3. [6] Let $G(V, E)$ be a graph. A mapping $f : V \to \{0, 1\}$ is called a *binary (vertex) labeling* of G and *f(v)* is called the label of the vertex v of G under *f*.
An *induced (edge) labeling* $f^* : E \to \{0, 1\}$ is defined by $f^*(uv) = |f(u) - f(v)|$ for each $uv \in E$.

Notation 1.4. [6] Given a binary labeling f of a graph $G(V, E)$, $v_f(i)$ or $v(i)$ and $e_f(i)$ or $e(i)$ denotes the number of vertices and edges respectively having label i, where $i = 0, 1$.

Definition 1.5. [4] A binary labeling f of a graph $G(V, E)$ is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.
The graph which admits cordial labeling is called a *cordial graph.*

Cahit [4], introduced the notion of cordial labeling as a weaker version of graceful labeling and harmonious labeling.
Another labeling, called prime labeling was originated by Entringer. It was first introduced by Tout et al [1] in their paper.

Definition 1.6. [1] Let $G(V, E)$ be a graph with n vertices. A bijection $f : V \to [n]$ is called a *prime labeling* if for each $uv \in E$, $gcd(f(u), f(v)) = 1$.
The graph which admits prime labeling is called a *prime graph.*

Combining the notion of prime labeling and cordial labeling, Sundaram, Ponraj, and Somasundram [9] introduced the notion of prime cordial labeling.

Definition 1.7. [9] Let $G(V, E)$ be a graph with n vertices. A bijection $f : V \to [n]$ is called a *prime cordial labeling* if induced edge labeling $f^* : E \to \{0, 1\}$ defined as

$$f^*(uv) = \begin{cases} 0, & \text{if } gcd(f(u), f(v)) \neq 1; \\ 1, & \text{if } gcd(f(u), f(v)) = 1, \end{cases}$$

satisfies $|e(0) - e(1)| \leq 1$. The graph which admits prime cordial labeling is called a *prime cordial graph.*

Ghodasara and Jena [3] proved that the graph G obtained by joining two copies of cycle with one chord and cycle with twin chord by a path of arbitrary length is prime cordial. They [7] also proved that the graph G obtained by joining two copies of Petersen graph, flower graph, fan graph, cycle with a triangle by a path of arbitrary length is prime cordial. J Babujee and Shobana [5] proved that $K_2 \Theta C_n(C_n)$ admits prime cordial labeling if $n \equiv 0, 2 (\text{mod } 3)$.

Analogous to prime labeling, Prajapati and Shah [10] introduced a new labeling called odd prime labeling.

Definition 1.8. [10] Let $G(V, E)$ be a graph with n vertices. A bijection $f : V \to O_n$ is called an *odd prime labeling* if for each $uv \in E$, $gcd(f(u), f(v)) = 1$.
A graph which admits odd prime labeling is called an *odd prime graph.*

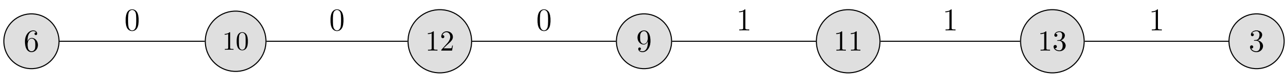
Prajapati and Shah [10] proved that graphs like path, star graph, complete bipartite graphs under certain conditions, wheel and wheel related graphs, Petersen graph $P(n, 2)$ are odd prime graphs. They [11] also proved that graphs obtained by duplicating each vertex by an edge and each edge by a vertex for path, star graph, cycle and wheel graph are all odd prime graphs.
Analogous to prime cordial labeling, we introduce the following definition on path P_k.

Definition 1.9. A path P_k of order $k \geq 3$ is said to have $P_k[a, b, n]$ labeling for positive integers $a < b < n$, if there exists a bijection $f : V(P_k) \to \{a, b, n, n+1, \ldots, n+k-3\}$ with $f(v) \in \{a, b\}$, when v is an end vertex and an induced edge labeling $f^* : E(P_k) \to \{0, 1\}$ defined as

$$f^*(uv) = \begin{cases} 0, & \text{if } gcd(f(u), f(v)) \neq 1; \\ 1, & \text{if } gcd(f(u), f(v)) = 1, \end{cases}$$

satisfying $|e(0) - e(1)| \leq 1$.

Example 1.10. P_7 satisfying $P_7[3, 6, 9]$:



Some number theoretical results required for this paper are noted here.

Result 1.11. For any odd integer n and positive integer k, $gcd(n, n + 2^k) = 1$.

Result 1.12. For any odd integer n, an odd prime p and positive integers k_1 and k_2 if $n \not\equiv 0 \pmod p$, then $gcd(n, n + 2^{k_1} \cdot p^{k_2}) = 1$.

2 Main Results

Theorem 2.1. *If $G_1(V_1, E_1)$ is an odd prime graph and $G_2(V_2, E_2)$ is a graph such that $|V_1| = |V_2|$ and $|E_1| = |E_2|$, then disjoint union of G_1 and G_2 is a prime cordial graph.*

Proof. Let $G_1(V_1, E_1)$ be an odd prime graph with $|V_1| = p$ and $|E_1| = q$. Thus, there exists a bijective function $f_1 : V_1 \to O_p$ such that for any $uv \in E_1$, $gcd(f_1(u), f_1(v)) = 1$.
Now, $G_2(V_2, E_2)$ is a graph with $|V_2| = |V_1| = p$ and $|E_2| = |E_1| = q$. Given any bijective function $f_2 : V_2 \to [2p] - O_p$, $f_2(u)$ is always an even number for any $u \in V_2$. Thus for any $uv \in E_2$, $gcd(f_2(u), f_2(v)) \neq 1$.
Now, consider a graph $G(V, E)$ obtained by the disjoint union of G_1 and G_2. Thus, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ with $|V| = 2p$ and $|E| = 2q$. Define a function $f : V \to [2p]$ as

$$f(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2. \end{cases}$$

Thus, the induced edge labeling, $f^* : E \to \{0, 1\}$ is obtained as:

$$f^*(e) = \begin{cases} 1, & \text{if } e \in E_1; \\ 0, & \text{if } e \in E_2. \end{cases}$$

This gives $e(1) = |E_1| = q$ and $e(0) = |E_2| = q$ establishing $|e(0) - e(1)| = 0 \leq 1$.
Hence, f admits prime cordial labeling on G and it is a prime cordial graph. □

Corollary 2.2. *If G is an odd prime graph, then the disjoint union of two copies of G is prime cordial.*

Theorem 2.3. *Path P_k satisfies $P_k[2, 3, n]$ labeling for all $n, k \geq 4$, $k \neq 5$.*

Proof. Let us denote the vertex set and edge set of P_k respectively as $V = \{v_i \,|\, i \in [k-1] \cup \{0\}\}$ and $E = \{e_i = v_{i-1}v_i \,|\, i \in [k-1]\}$. We prove the result taking different cases:

1. When n is odd; k is even, define:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n + 2i - 1, & \text{if } i \in \left[\frac{k}{2} - 1\right]; \\ n + 2i - k, & \text{if } i \in [k-2] - \left[\frac{k}{2} - 1\right]; \\ 3, & \text{if } i = k - 1. \end{cases} \tag{1}$$

Hence, the induced edge labeling except for $e_{\frac{k}{2}}$ and e_{k-1} is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k}{2} - 1\right]; \\ 1, & \text{if } i \in [k-2] - \left[\frac{k}{2}\right]. \end{cases}$$

For $e_{\frac{k}{2}}$ and e_{k-1}, we have:

$$f^*\left(e_{\frac{k}{2}}\right) = \begin{cases} 0, & \text{if } gcd(n+k-3, n) \neq 1; \\ 1, & \text{if } gcd(n+k-3, n) = 1. \end{cases} \text{ and } f^*(e_{k-1}) = \begin{cases} 0, & \text{if } gcd(n+k-4, 3) \neq 1; \\ 1, & \text{if } gcd(n+k-4, 3) = 1. \end{cases}$$

Hence, the following sub cases arise:

(a) If $gcd(n+k-3, n) = 1$ and $gcd(n+k-4, 3) = 1$, then $f^*\left(e_{\frac{k}{2}}\right) = 1$ and $f^*(e_{k-1}) = 1$. This gives $e(0) = \frac{k}{2} - 1$ and $e(1) = \frac{k}{2}$. Thus, vertex labeling f as given in (1) satisfies $|e(0) - e(1)| \leq 1$.

(b) If $gcd(n+k-3, n) = 1$ and $gcd(n+k-4, 3) \neq 1$, then $f^*\left(e_{\frac{k}{2}}\right) = 1$ and $f^*(e_{k-1}) = 0$. This gives $e(0) = \frac{k}{2}$ and $e(1) = \frac{k}{2} - 1$. Thus, vertex labeling f as given in (1) satisfies $|e(0) - e(1)| \leq 1$.

(c) If $gcd(n+k-3, n) \neq 1$ and $gcd(n+k-4, 3) = 1$, then $f^*(e_{\frac{k}{2}}) = 0$ and $f^*(e_{k-1}) = 1$. This gives $e(0) = \frac{k}{2}$ and $e(1) = \frac{k}{2} - 1$. Thus, vertex labeling f as given in (1) satisfies $|e(0) - e(1)| \leq 1$.

(d) If $gcd(n+k-3, n) \neq 1$ and $gcd(n+k-4, 3) \neq 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+2i-1, & \text{if } i \in \left[\frac{k}{2} - 1\right]; \\ n+2i-k, & \text{if } i \in [k-4] - \left[\frac{k}{2} - 1\right]; \\ n+k-4, & \text{if } i = k-3; \\ n+k-6, & \text{if } i = k-2; \\ 3, & \text{if } i = k-1. \end{cases} \tag{2}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k}{2}\right]; \\ 1, & \text{if } i \in [k-1] - \left[\frac{k}{2}\right], \end{cases}$$

which gives $e(0) = \frac{k}{2}$ and $e(1) = \frac{k}{2} - 1$. Thus, vertex labeling f as given in (2) satisfies $|e(0) - e(1)| \leq 1$.

2. When n is odd; k is odd, define:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n + 2i - 1, & \text{if } i \in \left[\dfrac{k-3}{2}\right]; \\ n + 2i - k + 1, & \text{if } i \in [k-2] - \left[\dfrac{k-3}{2}\right]; \\ 3, & \text{if } i = k - 1. \end{cases} \tag{3}$$

Hence, the induced edge labeling except for $e_{\frac{k-1}{2}}$ and e_{k-1} is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\dfrac{k-3}{2}\right]; \\ 1, & \text{if } i \in [k-2] - \left[\dfrac{k-1}{2}\right]. \end{cases}$$

For $e_{\frac{k-1}{2}}$ and e_{k-1}, we have:

$$f^*\left(e_{\frac{k-1}{2}}\right) = \begin{cases} 0, & \text{if } gcd(n+k-4, n) \neq 1; \\ 1, & \text{if } gcd(n+k-4, n) = 1. \end{cases} \text{ and } f^*(e_{k-1}) = \begin{cases} 0, & \text{if } gcd(n+k-3, 3) \neq 1; \\ 1, & \text{if } gcd(n+k-3, 3) = 1. \end{cases}$$

Hence, the following sub cases arise:

(a) If $gcd(n+k-4, n) \neq 1$ and $gcd(n+k-3, 3) = 1$ then $f^*\left(e_{\frac{k-1}{2}}\right) = 0$ and $f^*\left(e_{k-1}\right) = 1$. This gives $e(0) = e(1) = \dfrac{k-3}{2} + 1$. Thus, vertex labeling f as given in (3) satisfies $|e(0) - e(1)| \leq 1$.

(b) If $gcd(n+k-4, n) = 1$ and $gcd(n+k-3, 3) \neq 1$, then $f^*\left(e_{\frac{k-1}{2}}\right) = 1$ and $f^*\left(e_{k-1}\right) = 0$. This gives $e(0) = e(1) = \dfrac{k-3}{2} + 1$. Thus, vertex labeling f as given in (3) satisfies $|e(0) - e(1)| \leq 1$.

(c) If $gcd(n+k-4, n) \neq 1$ and $gcd(n+k-3, 3) \neq 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n + 2i - 1, & \text{if } i \in \left[\dfrac{k-3}{2}\right]; \\ n + 2i - k + 1, & \text{if } i \in [k-4] - \left[\dfrac{k-3}{2}\right]; \\ n + k - 3, & \text{if } i = k - 3; \\ n + k - 5, & \text{if } i = k - 2; \\ 3, & \text{if } i = k - 1. \end{cases} \tag{4}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\dfrac{k-1}{2}\right]; \\ 1, & \text{if } i \in [k-1] - \left[\dfrac{k-1}{2}\right], \end{cases}$$

which gives $e(0) = e(1) = \dfrac{k-3}{2} + 1$. Thus, vertex labeling f as given in (4) satisfies $|e(0) - e(1)| \leq 1$.

(d) If $gcd(n+k-4, n) = 1$ and $gcd(n+k-3, 3) = 1$, then,

(i) for $gcd(n+k-5, 3) \neq 1$, vertex labeling f as given in (4) gives induced edge labeling as:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k-3}{2}\right] \cup \{k-1\}; \\ 1, & \text{if } i \in [k-2] - \left[\frac{k-3}{2}\right], \end{cases}$$

which gives $e(0) = e(1) = \frac{k-3}{2}+1$. Thus, vertex labeling f as given in (4) satisfies $|e(0) - e(1)| \leq 1$.

(ii) for $gcd(n+k-7, 3) \neq 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+2i-1, & \text{if } i \in \left[\frac{k-3}{2}\right]; \\ n+2i-k+1, & \text{if } i \in [k-5] - \left[\frac{k-3}{2}\right]; \\ n+k-5, & \text{if } i = k-4; \\ n+k-3, & \text{if } i = k-3; \\ n+k-7, & \text{if } i = k-2; \\ 3, & \text{if } i = k-1. \end{cases} \tag{5}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k-3}{2}\right] \cup \{k-1\}; \\ 1, & \text{if } i \in [k-2] - \left[\frac{k-3}{2}\right], \end{cases}$$

which gives $e(0) = e(1) = \frac{k-3}{2}+1$. Thus, vertex labeling f as given in (5) satisfies $|e(0) - e(1)| \leq 1$.

3. When n is even; k is odd, define:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+2i-2, & \text{if } i \in \left[\frac{k-1}{2}\right]; \\ n+2i-k+1, & \text{if } i \in [k-2] - \left[\frac{k-1}{2}\right]; \\ 3, & \text{if } i = k-1. \end{cases} \tag{6}$$

Hence, the induced edge labeling except for $e_{\frac{k+1}{2}}$ and e_{k-1} is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k-1}{2}\right]; \\ 1, & \text{if } i \in [k-2] - \left[\frac{k+1}{2}\right]. \end{cases}$$

For $e_{\frac{k+1}{2}}$ and e_{k-1}, we have:

$$f^*\left(e_{\frac{k+1}{2}}\right) = \begin{cases} 0, & \text{if } gcd(n+k-3, n+1) \neq 1; \\ 1, & \text{if } gcd(n+k-3, n+1) = 1. \end{cases} \text{ and } f^*(e_{k-1}) = \begin{cases} 0, & \text{if } gcd(n+k-4, 3) \neq 1; \\ 1, & \text{if } gcd(n+k-4, 3) = 1. \end{cases}$$

Hence, the following sub cases arise:

(a) If $gcd(n+k-3, n+1) = 1$ and $gcd(n+k-4, 3) = 1$, then $f^*\left(e_{\frac{k+1}{2}}\right) = 1$ and $f^*(e_{k-1}) = 1$. This gives $e(0) = e(1) = \frac{k-1}{2}$. Thus, vertex labeling f as given in (6) satisfies $|e(0) - e(1)| \leq 1$.

(b) If $gcd(n+k-3, n+1) = 1$ and $gcd(n+k-4, 3) \neq 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+2i-2, & \text{if } i \in \left[\frac{k-1}{2}\right]; \\ n+2i-k+1, & \text{if } i \in [k-4] - \left[\frac{k-1}{2}\right]; \\ n+k-4, & \text{if } i = k-3; \\ n+k-6, & \text{if } i = k-2; \\ 3, & \text{if } i = k-1. \end{cases} \tag{7}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k-1}{2}\right]; \\ 1, & \text{if } i \in [k-1] - \left[\frac{k-1}{2}\right], \end{cases}$$

which gives $e(0) = e(1) = \frac{k-1}{2}$. Thus, vertex labeling f as given in (7) satisfies $|e(0) - e(1)| \leq 1$.

(c) If $gcd(n+k-3, n+1) \neq 1$ and $gcd(n+k-4, 3) = 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+k-3, & \text{if } i = 1; \\ n+2i-2, & \text{if } i \in \left[\frac{k-3}{2}\right] - \{1\}; \\ n, & \text{if } i = \frac{k-1}{2}; \\ n+2i-k+1, & \text{if } i \in [k-2] - \left[\frac{k-1}{2}\right]; \\ 3, & \text{if } i = k-1. \end{cases} \tag{8}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k-1}{2}\right]; \\ 1, & \text{if } i \in [k-1] - \left[\frac{k-1}{2}\right], \end{cases}$$

which gives $e(0) = e(1) = \frac{k-1}{2}$. Thus, vertex labeling f as given in (8) satisfies $|e(0) - e(1)| \leq 1$.

(d) If $gcd(n+k-3,n+1) \neq 1$ and $gcd(n+k-4,3) \neq 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+k-3, & \text{if } i = 1; \\ n+2i-2, & \text{if } i \in \left[\dfrac{k-3}{2}\right] - \{1\}; \\ n, & \text{if } i = \dfrac{k-1}{2}; \\ n+2i-k+1, & \text{if } i \in [k-4] - \left[\dfrac{k-1}{2}\right]; \\ n+k-4, & \text{if } i = k-3; \\ n+k-6, & \text{if } i = k-2; \\ 3, & \text{if } i = k-1. \end{cases} \tag{9}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\dfrac{k-1}{2}\right]; \\ 1, & \text{if } i \in [k-1] - \left[\dfrac{k-1}{2}\right], \end{cases}$$

which gives $e(0) = e(1) = \dfrac{k-1}{2}$. Thus, vertex labeling f as given in (9) satisfies $|e(0) - e(1)| \leq 1$.

4. When n is even; k is even, define:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+2i-2, & \text{if } i \in \left[\dfrac{k}{2} - 1\right]; \\ n+2i-k, & \text{if } i \in [k-2] - \left[\dfrac{k}{2} - 1\right]; \\ 3, & \text{if } i = k-1. \end{cases} \tag{10}$$

Hence, the induced edge labeling except for $e_{\frac{k}{2}}$ and e_{k-1} is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\dfrac{k}{2} - 1\right]; \\ 1, & \text{if } i \in [k-2] - \left[\dfrac{k}{2}\right]. \end{cases}$$

For $e_{\frac{k}{2}}$ and e_{k-1}, we have:

$f^*\left(e_{\frac{k}{2}}\right) = \begin{cases} 0, & \text{if } gcd(n+k-4,n+1) \neq 1; \\ 1, & \text{if } gcd(n+k-4,n+1) = 1. \end{cases}$ and $f^*(e_{k-1}) = \begin{cases} 0, & \text{if } gcd(n+k-3,3) \neq 1; \\ 1, & \text{if } gcd(n+k-3,3) = 1. \end{cases}$

Hence, the following subcases arise:

(a) If $gcd(n+k-4,n+1) = 1$ and $gcd(n+k-3,3) = 1$, then $f^*\left(e_{\frac{k}{2}}\right) = 1$ and $f^*(e_{k-1}) = 1$. This gives $e(0) = \dfrac{k}{2} - 1$ and $e(1) = \dfrac{k}{2}$. Thus, vertex labeling f as given in (10) satisfies $|e(0) - e(1)| \leq 1$.

(b) If $gcd(n+k-4, n+1) = 1$ and $gcd(n+k-3, 3) \neq 1$, then $f^*\left(e_{\frac{k}{2}}\right) = 1$ and $f^*(e_{k-1}) = 0$. This gives $e(0) = \frac{k}{2}$ and $e(1) = \frac{k}{2} - 1$. Thus, vertex labeling f as given in (10) satisfies $|e(0) - e(1)| \leq 1$.

(c) If $gcd(n+k-4, n+1) \neq 1$ and $gcd(n+k-3, 3) = 1$, then $f^*(e_{\frac{k}{2}}) = 0$ and $f^*(e_{k-1}) = 1$. This gives $e(0) = \frac{k}{2}$ and $e(1) = \frac{k}{2} - 1$. Thus, vertex labeling f as given in (10) satisfies $|e(0) - e(1)| \leq 1$.

(d) If $gcd(n+k-4, n+1) \neq 1$ and $gcd(n+k-3, 3) \neq 1$, define f as:

$$f(v_i) = \begin{cases} 2, & \text{if } i = 0; \\ n+2i-2, & \text{if } i \in \left[\frac{k}{2} - 1\right]; \\ n+2i-k, & \text{if } i \in [k-4] - \left[\frac{k}{2} - 1\right]; \\ n+k-3, & \text{if } i = k-3; \\ n+k-5, & \text{if } i = k-2; \\ 3, & \text{if } i = k-1. \end{cases} \tag{11}$$

Thus the induced edge labeling is given by:

$$f^*(e_i) = \begin{cases} 0, & \text{if } i \in \left[\frac{k}{2}\right]; \\ 1, & \text{if } i \in [k-1] - \left[\frac{k}{2}\right], \end{cases}$$

which gives $e(0) = \frac{k}{2}$ and $e(1) = \frac{k}{2} - 1$.
Thus, vertex labeling f as given in (11) satisfies $|e(0) - e(1)| \leq 1$.

Thus, from all the cases, we see that P_k satisfies $P_k[2, 3, n]$ labeling for $n, k \geq 4$ and $k \neq 5$. □

Corollary 2.4. *Path P_k satisfies $P_k[2^m, 3, n]$ labeling for all $k \geq 4$, $k \neq 5$; $n > max\{3, 2^m\}$, $m \in N$.*

Note. Henceforth, throughout the paper unless mentioned explicitly we follow the given assumptions:

1. $G(V, E)$ is an odd prime graph of order n. $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are two copies of G with $V_1 = \{v_{1i} \,|\, i \in [n]\}$ and $V_2 = \{v_{2i} \,|\, i \in [n]\}$ where v_{1i} and v_{2i} are corresponding vertices.

2. H is a graph obtained by identifying each end vertex of a path P_k with corresponding vertices of G_1 and G_2.

Theorem 2.5. *Let G be an odd prime graph of order n and H be a graph obtained by joining any pair of corresponding vertices of two copies G_1 and G_2 of G by an edge. Then H is a prime cordial graph.*

Proof. $G_1(V_1, E_1) = G$ is an odd prime graph. Thus, there exists a bijective function $f_1 : V_1 \to O_n$ such that for any $v_{1i}v_{1j} \in E_1, i \neq j$, $gcd(f_1(v_{1i}), f_1(v_{1j})) = 1$. For $G_2(V_2, E_2)$, define a bijective function $f_2 : V_2 \to [2n] - O_n$ as $f_2(v_{2i}) = f_1(v_{1i}) + 1$ which is always an even number. Hence, for any $v_{2i}v_{2j} \in E_2, i \neq j$, $gcd(f_2(v_{2i}), f_1(v_{2j})) \neq 1$.
Now, H is the graph obtained by adding an edge to a pair of corresponding vertices of graphs G_1

and G_2. Without loss of generality, assume that corresponding vertices v_{1l} and v_{2l} are joined by an edge. Thus, graph H has vertex set $V(H) = V_1 \cup V_2$ with $|V(H)| = 2n$.
Define a function: $h : V(H) \to [2n]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2. \end{cases}$$

This, gives $h(v_{1l}) = f_1(v_{1l})$ and $h(v_{2l}) = f_2(v_{2l}) = f_1(v_{1l}) + 1$. Hence, $h^*(v_{1l}v_{2l}) = 1$ and the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{v_{1l}v_{2l}\}; \\ 0, & \text{if } e \in E_2. \end{cases}$$

Hence we see that $e(0) = n$ and $e(1) = n + 1$, thus establishing $|e(1) - e(0)| \leq 1$.
Thus H is a prime cordial graph. □

Theorem 2.6. *Let G be a graph of order 2 and H be a graph obtained by identifying each end vertex of a path P_3 with corresponding vertices of two copies of G. Then H is not a prime cordial graph.*

Proof. Graph G is of order 2. Thus, the only possibility for graph G is the complete graph K_2. With path P_3 the only possiblity for graph H is a path P_5 and it is not a prime cordial graph [9]. □

Theorem 2.7. *Let G be a graph of order n and H be a graph obtained by identifying each end vertex of a path P_k with corresponding vertices of two copies of G. Then H is a prime cordial graph if:*

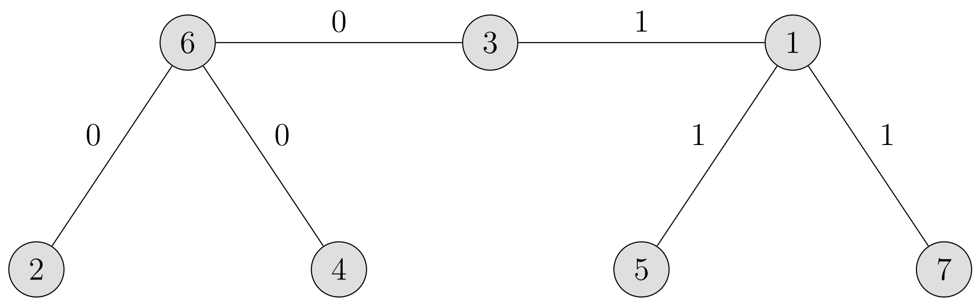
1. *$k = 3$ and $n = 3$;*
2. *$k = 5$ and $n = 2$.*

Proof. 1. For $k = 3$ and $n = 3$, the graph G is of order 3 and the path is P_3. In this case, possibilities for a connected simple graph G are: P_3 and K_3 . Thus, when end vertices of P_3 are identified with corresponding vertices of two copies of G, graph H has the following two possibilities:

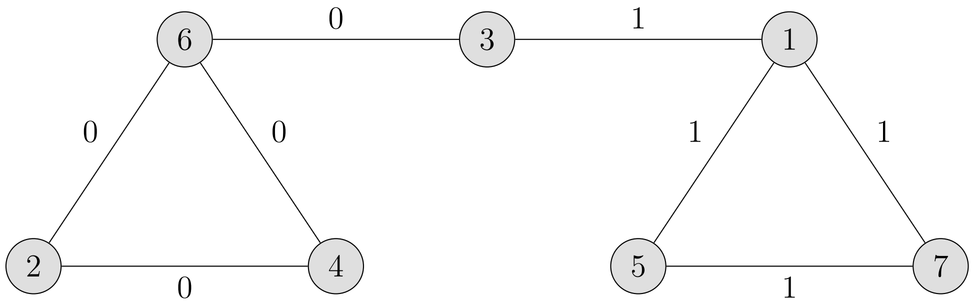
(a) if $G = P_3$, then:

(i) H becomes a path P_7 when end vertices of P_3 are identified with corresponding end vertices of G_1 and G_2 and it is a prime cordial graph [9].

(ii) H becomes a graph as shown in figure below when end vertices of P_3 are identified with corresponding non-pendant vertices of G_1 and G_2 and it is also a prime cordial graph.



(b) if $G = K_3$ then the labeling shown below gives prime cordial labeling for H.



2. For $k = 5$ and $n = 2$, graph G is of order 2 and the path is P_5. The only possibility for graph G is the complete graph K_2. Thus, when end vertices of P_5 are identified with corresponding vertices of G_1 and G_2, we obtain H as a path P_7 and it is a prime cordial graph [9].

Thus in both the cases, we obtain H as a prime cordial graph. □

Theorem 2.8. *Let G be an odd prime graph and H be a graph obtained by identifying each end vertex of a path P_k with corresponding vertices of each of the two copies G_1 and G_2 of G. Then H is a prime cordial graph, when one end vertex of P_k is identified with a vertex in G_1 having label 3 if:*

1. $k \geq 4$, $k \neq 5$ *and* $n \geq 2$;
2. $k = 3$ *and* $n \geq 4$;
3. $k = 5$ *and* $n \geq 3$.

Proof. $G_1(V_1, E_1) = G$ is an odd prime graph. Thus, there exists a bijective function $f_1 : V_1 \to O_n$ such that for any $v_{1i}v_{1j} \in E_1, i \neq j$, $gcd(f_1(v_{1i}), f_1(v_{1j})) = 1$.
For $G_2(V_2, E_2)$, define a bijective function $f_2 : V_2 \to [2n] - O_n$ as $f_2(v_{2i}) = f_1(v_{1i}) + 1$ which is always an even number. Hence, for any $v_{2i}v_{2j} \in E_2, i \neq j$, $gcd(f_2(v_{2i}), f_1(v_{2j})) \neq 1$.
Let $w_0, w_1, \ldots, w_{k-1}$ be consecutive vertices of P_k with w_0 and w_{k-1} as its end vertices. Identify w_0 and w_{k-1} with corresponding vertices of G_2 and G_1 respectively such that $w_{k-1} = v_{1t} \in G_1$ for some $t \in [n]$. As one end vertex of the path is identified with the vertex in G_1 having label 3, we have: $f_1(w_{k-1}) = f_1(v_{1t}) = 3$. We prove different cases as follows:

1. $k \geq 4$ and $k \neq 5$.
By corollary 2.4, P_k satisfies the labeling $P_k[3, 4, 2n + 1]$. This is possible as $n \geq 2$.
Thus, there exists a bijective function: $f_3 : V(P_k) \to \{3, 4, 2n + 1, 2n + 2, \ldots, 2n + k - 2\}$ with $f_3(w_{k-1}) = 3$ and $f_3(w_0) = 4$ satisfying $|e_{f_3}(1) - e_{f_3}(0)| \leq 1$. Thus we have: $f_3(w_{k-1}) = f_1(v_{1t}) = 3$ and $f_3(w_0) = f_2(v_{2t}) = 4$. Now H is the graph obtained by identifying the end vertices w_{k-1} and w_0 of P_k with corresponding vertices v_{1t} and v_{2t} respectively. Thus H has vertex set as $V(H) = V_1 \cup V_2 \cup V(P_k)$ with $|V(H)| = 2n + k - 2$ and we define: $h : V(H) \to [2n + k - 2]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2; \\ f_3(u), & \text{if } u \in V(P_k). \end{cases}$$

Hence we see that $e_{f_1}(0) = e_{f_2}(1) = n$ and $|e_{f_3}(1) - e_{f_3}(0)| \leq 1$, establishing $|e_h(1) - e_h(0)| \leq 1$. Thus H is a prime cordial graph.

2. $k = 3$ and $n \geq 4$.
Here H is a graph obtained by identifying the end vertices of P_3 with corresponding vertices of G_1 and G_2. Consecutive vertices of P_3 are w_0, w_1, w_2 with $v_{1t} = w_2$ and $v_{2t} = w_0$. Now H has vertex set $V(H) = V_1 \cup V_2 \cup \{w_1\}$ with $|V(H)| = 2n + 1$.
We prove the result with two cases:

   (a) If $2n + 1 \equiv 0(\text{mod } 3)$, define a function: $h : V(H) \to [2n + 1]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2; \\ 2n + 1, & \text{if } u = w_1. \end{cases}$$

   Since $2n + 1 \equiv 0(\text{mod } 3)$, $h^*(w_0w_1) = 1$ and $h^*(w_1w_2) = 0$ and hence the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_0w_1\}; \\ 0, & \text{if } e \in E_2 \cup \{w_1w_2\}. \end{cases}$$

   (b) If $2n + 1 \not\equiv 0(\text{mod } 3)$,
   we again divide this into two sub cases:

      (i) If $2n + 1$ is prime, $gcd(2n + 1, i) = 1$, for all $i \in O_n$.
      Without loss of generality, assume that $f(v_{1s}) = 9$ for some $s \neq t \in [n]$.
      This is possible as $n \geq 4$. Now, define $h : V(H) \to [2n + 1]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1 - \{v_{1s}\}; \\ 2n + 1, & \text{if } u = v_{1s}; \\ f_2(u), & \text{if } u \in V_2; \\ 9, & \text{if } u = w_1. \end{cases}$$

      Thus, we have $h^*(w_0w_1) = 1$ and $h^*(w_1w_2) = 0$ and the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_0w_1\}; \\ 0, & \text{if } e \in E_2 \cup \{w_1w_2\}. \end{cases}$$

      (ii) If $2n + 1$ is not prime, there exists a prime $p < n$ such that $p \,|\, 2n + 1$. Thus, $gcd(2n + 1, 2p) \neq 1$. Again as $2p < 2n$, there exists $q \neq t \in [n]$ such that $f_2(v_{2q}) = 2p$. Now, define $h : V(H) \to [2n + 1]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2 - \{v_{2t}, v_{2q}\}; \\ 4, & \text{if } u = v_{2q}; \\ 2p, & \text{if } u = v_{2t} = w_0; \\ 2n + 1, & \text{if } u = w_1. \end{cases}$$

      Thus, we have $h^*(w_0w_1) = 0$ and $h^*(w_1w_2) = 1$ and the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_1w_2\}; \\ 0, & \text{if } e \in E_2 \cup \{w_0w_1\}. \end{cases}$$

Thus from all the cases, the induced edge labeling gives $e_h(0) = e_h(1) = n+1$, thus establishing $|e_h(1) - e_h(0)| \leq 1$. Thus H is a prime cordial graph.

3. $k = 5$ and $n \geq 3$.
Here H is a graph obtained by identifying the end vertices of P_5 with corresponding vertices of G_1 and G_2. Consecutive vertices of P_5 are w_0, w_1, w_2, w_3, w_4 with $v_{1t} = w_4$ and $v_{2t} = w_0$. Now H has vertex set $V(H) = V_1 \cup V_2 \cup \{w_1, w_2, w_3\}$ with $|V(H)| = 2n + 3$.
We divide the proof into following cases and its subcases:

   (a) $n \not\equiv 2 (\text{mod } 3)$. This is further divided into two subcases:

      (i) $n \equiv 0(\text{mod } 3) \implies 2n + 3 \equiv 0(\text{mod } 3)$. Here, define $h : V(H) \to [2n + 3]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2; \\ 2n+2, & \text{if } u = w_1; \\ 2n+1, & \text{if } u = w_2; \\ 2n+3, & \text{if } u = w_3. \end{cases}$$

      (ii) $n \equiv 1(\text{mod } 3) \implies 2n + 1 \equiv 0(\text{mod } 3)$. Here, define $h : V(H) \to [2n + 3]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2; \\ 2n+2, & \text{if } u = w_1; \\ 2n+3, & \text{if } u = w_2; \\ 2n+1, & \text{if } u = w_3. \end{cases}$$

   In both the cases, the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_1w_2, w_2w_3\}; \\ 0, & \text{if } e \in E_2 \cup \{w_0w_1, w_3w_4\}. \end{cases}$$

   (b) $n \equiv 2(\text{mod } 3)$. This is further divided into two subcases:

      (i) $2n + 3$ is prime $\implies gcd(2n + 3, i) = 1$ for each $i \in O_n$.
      Without loss of generality, assume that $f(v_{1s}) = 9$ for some $s \neq t \in [n]$.
      This is possible as $n \geq 3$. Define $h : V(H) \to [2n + 3]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1 - \{v_{1s}\}; \\ 2n+3, & \text{if } u = v_{1s}; \\ f_2(u), & \text{if } u \in V_2; \\ 2n+2, & \text{if } u = w_1; \\ 2n+1, & \text{if } u = w_2; \\ 9, & \text{if } u = w_3. \end{cases}$$

      In this case, the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_1w_2, w_2w_3\}; \\ 0, & \text{if } e \in E_2 \cup \{w_0w_1, w_3w_4\}. \end{cases}$$

      (ii) $2n + 3$ is not prime. Again we have sub-subcases as follows:

         (A) $2n + 1$ is prime $\implies gcd(2n + 1, i) = 1$ for each $i \in O_n$.
         As $2n + 3$ is not prime, there exists a prime $p \in O_n$ such that $p \,|\, 2n + 3$. Thus

$gcd(2n+3, p) \neq 1$ and there exists some $v_{1l} \in V_1$, $l \neq t$ such that $f(v_{1l}) = p$. Define $h : V(H) \to [2n+3]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1 - \{v_{1l}\}; \\ 2n+1, & \text{if } u = v_{1l}; \\ f_2(u), & \text{if } u \in V_2; \\ 2n+2, & \text{if } u = w_1; \\ p, & \text{if } u = w_2; \\ 2n+3, & \text{if } u = w_3. \end{cases}$$

In this case, the induced edge labeling is given by:

$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_1w_2, w_3w_4\}; \\ 0, & \text{if } e \in E_2 \cup \{w_0w_1, w_2w_3\}. \end{cases}$$

(B) $2n+1$ is not prime $\implies$ there exists a prime $q < n$ such that $q \mid 2n+1$. Thus $gcd(2n+1, 2q) \neq 1$. As $2q < 2n$, there exists a vertex $v_{2m} \in G_2, m \neq t$ with $f_2(v_{2m}) = 2q$. Now, define the function, $h : V(H) \to [2n+3]$ as:

$$h(u) = \begin{cases} f_1(u), & \text{if } u \in V_1; \\ f_2(u), & \text{if } u \in V_2 - \{v_{2m}\}; \\ 2n+2, & \text{if } u = v_{2m}; \\ 2q, & \text{if } u = w_1; \\ 2n+1, & \text{if } u = w_2; \\ 2n+3, & \text{if } u = w_3. \end{cases}$$

In this case, the induced edge labeling is given by:

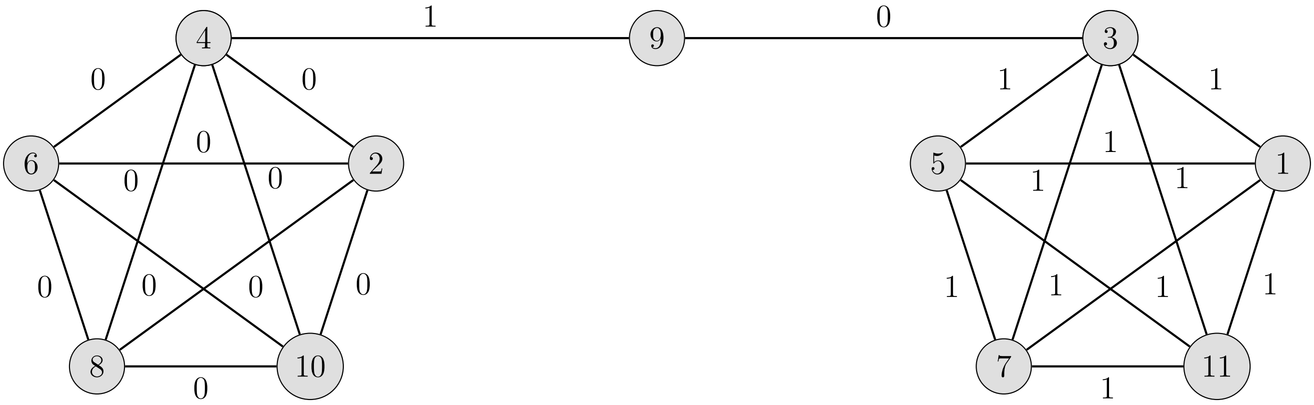
$$h^*(e) = \begin{cases} 1, & \text{if } e \in E_1 \cup \{w_2w_3, w_3w_4\}; \\ 0, & \text{if } e \in E_2 \cup \{w_0w_1, w_1w_2\}. \end{cases}$$

Thus from all the sub-cases, the induced edge labeling gives $e_h(0) = e_h(1) = n+2$, thus establishing $|e_h(1) - e_h(0)| \leq 1$. Thus H is a prime cordial graph.

Thus in both the cases, we see that H becomes prime cordial graph. □

From theorems 2.5, 2.6, 2.7 and 2.8 we see that graph H obtained by identifying each end vertex of a path P_k, with corresponding vertices of two copies of an odd prime graph G of order n is prime cordial except for $n = 2$ and $k = 3$.
The above conditions are only sufficient conditions. Although G may not be an odd prime graph, H may still be a prime cordial graph as seen in the example below.

Example 2.9. Graph obtained by identifying end vertices of P_3 with consecutive vertices of two copies of the complete graph K_5 is an odd prime graph.



3 Conclusion

This paper discussed the conditions on the graph obtained by identifying the end vertices of a path P_k with corresponding vertices of two copies of graph G of order n. It is observed that if G is an odd prime graph, then except for $n = 2$ and $k = 3$ all graphs are prime cordial graphs. Graph G being an odd prime graph is a sufficient condition and not necessary.

4 Open Problems

This paper only deals with the graphs that are odd prime graphs. The paper also limits that one end vertex of the path is identified with the vertex having label 3. Thus as open problem, one can investigate for the prime cordial labeling of graph families considering any pair of consecutive vertices and/or G not being an odd prime graph.

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