$(1, N)$-Arithmetic Labelling of Arbitrary Supersubdivision of Paths,Cycles and Stars

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Abstract

A (p, q)-graph G is said to be $(1, N)$-Arithmetic if there is a function ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$. In this paper we prove that the arbitrary supersubdivisions of paths, cycles and stars are $(1, N)$-Arithmetic Labelling for all positive integers N.

Keywords : $(1, N)$-Arithmetic Labelling, Supersubdivisions , Arbitrary supersubdivision, paths, cycles and stars.

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1 Introduction and Definition

S. W. Golomb [5] introduced graceful labelling. Odd gracefulness was introduced by R. B. Gnanajothi [6]. A graph G with q edges is said to be odd graceful if there is an injection f from the vertices of G to the set $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. C. Sekar [10] introduced one modulo three graceful labelling. A graph G with q edges is said to be one modulo three graceful if there is an injection f from the vertices of G to the set $\{0, 1, 3, 4, 6, 7, \ldots, 3(q-1), 3q-2\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 4, 7, \ldots, 3q-2\}$. V.Ramachandran and C. Sekar introduced one modulo N graceful where N is any positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful.

B. D. Acharya and S. M. Hegde [2] introduced (k, d)- arithmetic graphs. A (p, q)- graph G is said to be (k, d)- arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k+d, k+2d, \ldots, k+(q-1)d$.

A (p, q)-graph G is said to be $(1, N)$-arithmetic if there is a function $\phi : V(G) \rightarrow \{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$. In this situation the induced mapping ϕ^* to the edges is given by $\phi^*(uv) = \phi(u) + \phi(v)$. If the values of $\phi(u) + \phi(v)$ are $1, N+1, 2N+1, \ldots, N(q-1)+1$ all distinct, then we call the labelling of vertices as $(1, N)$- arithmetic labelling. In case if the induced mapping ϕ^* is defined as $\phi^*(uv) = |\phi(u) - \phi(v)|$ and if the resulting edge labels are are distinct and equal to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$. We call it as one modulo N graceful. In this paper we prove that $(1, N)$-Arithmetic Labelling of arbitrary supersubdivisions of paths, cycles and stars are for all positive integers N.

2 Main Results

Definition 2.1. *A graph G with q edges is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to* $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-$

$1), N(q-1)+1\}$ *in such a way that (i)* ϕ *is* $1-1$ *(ii)* ϕ *induces a bijection* ϕ^* *from the edge set of* G *to* $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ *where* $\phi^*(uv) = |\phi(u)-\phi(v)|$.

Definition 2.2. *A* (p,q) *-graph* G *is said to be* $(1,N)$ *-Arithmetic if there is a function* ϕ *from the vertex set* $V(G)$ *to* $\{0,1,N,(N+1),2N,(2N+1),\ldots,N(q-1),N(q-1)+1\}$ *so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression* $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$.

Definition 2.3. *A* (p,q) *- graph* G *is said to be* (k,d) *- arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression* $k, k+d, k+2d, \ldots, k+(q-1)d$.

Definition 2.4. *In the complete bipartite graph* $K_{2,m}$ *we call the part consisting of two vertices, the 2-vertices part of* $K_{2,m}$ *and the part consisting of* m *vertices the* m *-vertices part of* $K_{2,m}$*. Let* G *be a graph with* p *vertices and* q *edges. A graph* H *is said to be a supersubdivision of* G *if* H *is obtained by replacing every edge* e_i *of* G *by the complete bipartite graph* $K_{2,m}$ *for some positive integer* m *in such a way that the ends of* e_i *are merged with the two vertices part of* $K_{2,m}$ *after removing the edge* e_i *from* G*.* H *is denoted by* $SS(G)$.

Definition 2.5. *A supersubdivision* H *of a graph* G *is said to be an arbitrary supersubdivision of the graph* G *if every edge of* G *is replaced by an arbitrary* $K_{2,m}$ *(*m *may vary for each edge arbitrarily).* H *is denoted by* $ASS(G)$.

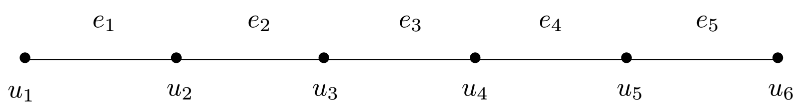
Definition 2.6. *A star* S_n *with* n *spokes is given by* (V,E) *where* $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ *and* $E(S_n) = \{v_0 v_i / i = 1, 2 \ldots, n\}$. v_0 *is called the centre of the star.*

Definition 2.7. *Cycle* C_n *with* n *points is a graph given by* (V,E) *where* $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ *and* $E(C_n) = \{v_1v_2, v_2v_{3,} \ldots, v_{n-1}v_n, v_nv_1\}$.

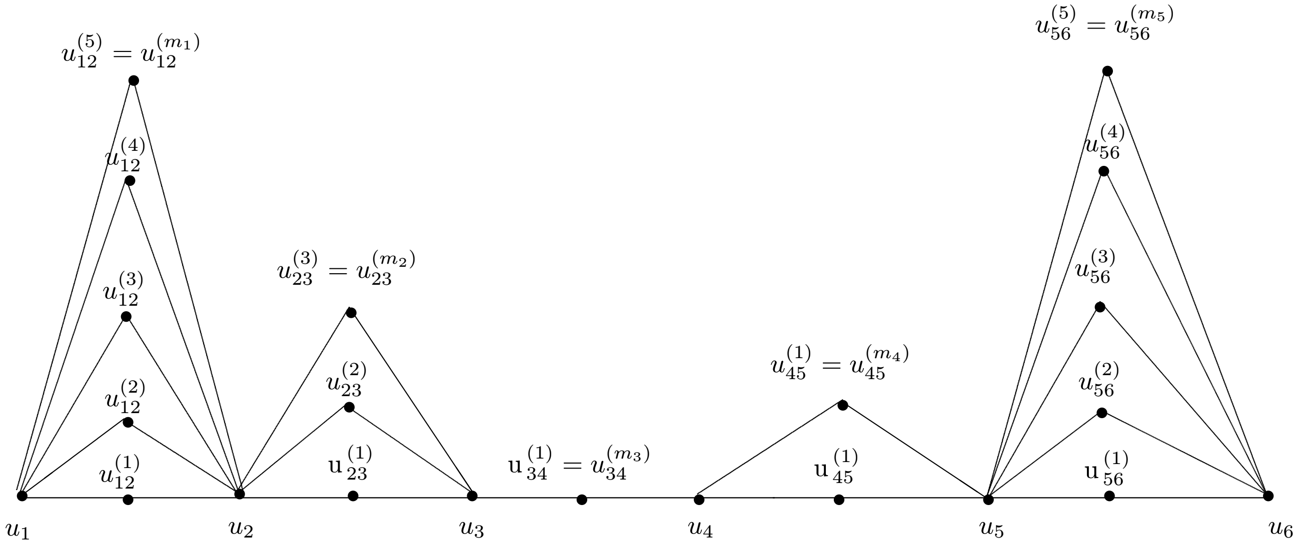
Theorem 2.8. *Arbitrary supersubdivisions of paths are* $(1,N)$ *-Arithmetic labelling for every positive integer* N *for every positive integer* $N>1$.

Proof. Let P_n be a path with successive vertices $u_1, u_2, u_3, \ldots\ldots u_n$ and let e_i $(1 \le i \le n-1)$ denote the edge u_iu_{i+1} of P_n. Let H be an arbitrary supersubdivision of the path P_n where each edge e_i of P_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer . We observe that H has

$M = 2(m_1 + m_2 + \ldots\ldots + m_{n-1})$ *edges.*



Path P_6



An arbitrary supersubdivision of P_6

Define

$\phi(u_i) = N(i-1), i = 1, 2, 3, \ldots, n$

For $k = 1, 2, 3, \ldots, m_i$

$$\phi(u_{i,i+1}^{(k)}) = \begin{cases} 2N(k-1)+1 & \text{if } i = 1 \\ N(2k+i-1)+2N(m_1+m_2+\cdots+m_{i-1}-i)+1 & \text{if } i = 2, 3, \ldots n-1 \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \leq i \leq n-1$ .Therefore the vertices of each K_{2,m_i} , 1 $\leq i \leq n-1$ in the arbitrary supersubdivision H of P_n have distinct labels and also the edges of each K_{2,m_i} , 1 $\leq i \leq n-1$ in the arbitrary supersubdivision graph H of P_n have distinct labels.

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$.

Also the induced mapping ϕ^* to the edges is given by $\phi^*(uv) = \phi(u)+\phi(v)$. If the values of $\phi(u)+\phi(v)$ are $1, N+1, 2N+1, \ldots, N(q-1)+1$ all distinct. Hence H is $(1, N)$-Arithmetic labelling for every positive integer $N > 1$.

Clearly ϕ defines a $(1, N)$-Arithmetic labelling of arbitrary supersubdivision of path P_n.

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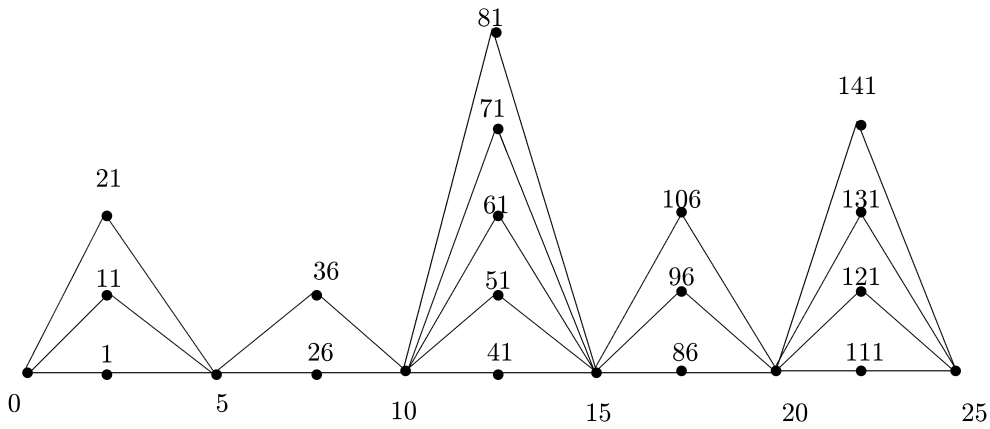
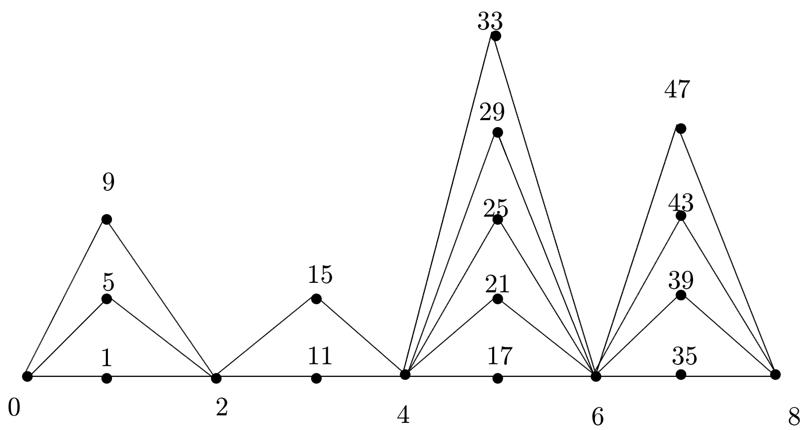
Example 2.9. $(1,5)$-*Arithmetic Labelling of* $ASS(P_6)$

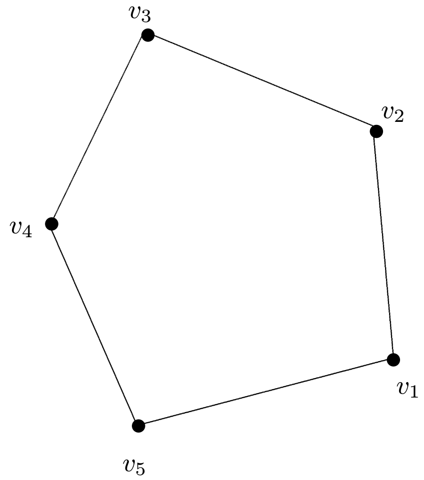


Example 2.10. $(1,2)$-*Arithmetic Labelling of* $ASS(P_5)$

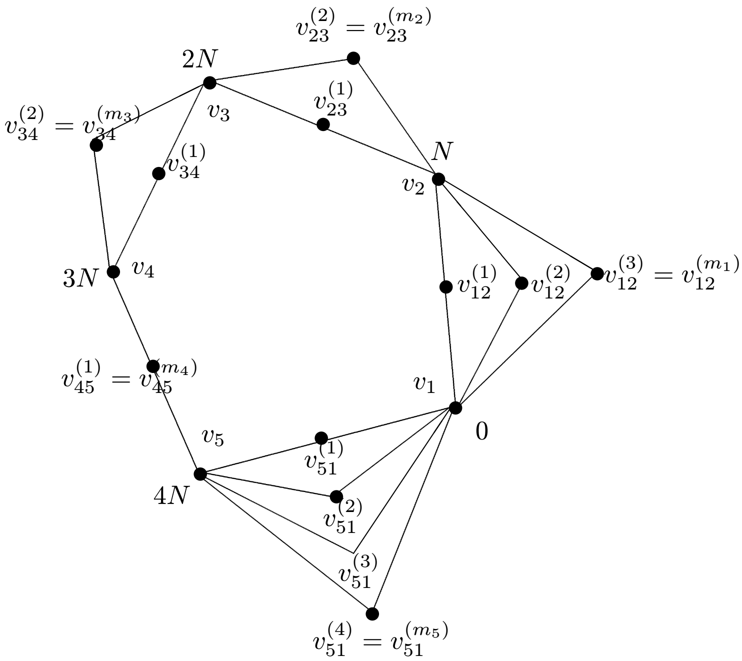


Theorem 2.11. *For any any $n \geq 3$, there exists an arbitrary supersubdivision of C_n which is $(1, N)$-Arithmetic Labelling for every positive integer $N > 1$.*

Proof. Let C_n be a cycle with consecutive vertices $v_1, v_2, v_3, ...v_n$. Let G be a supersubdivision of a cycle C_n where each edge e_i of C_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that G has $M = 2(m_1 + m_2 + ... + m_n)$ *edges.* Here the edge $v_{n-1}v_1$ is replaced by $K_{2,n-1}$ for the construction of arbitrary supersubdivision of C_n



$Cycle C_n$



An arbitrary Supersubdivision of C_5

Define

$\phi(v_i) = N(i-1), i = 1, 2, 3, \ldots, n$

For $k = 1, 2, 3, \ldots, m_i$

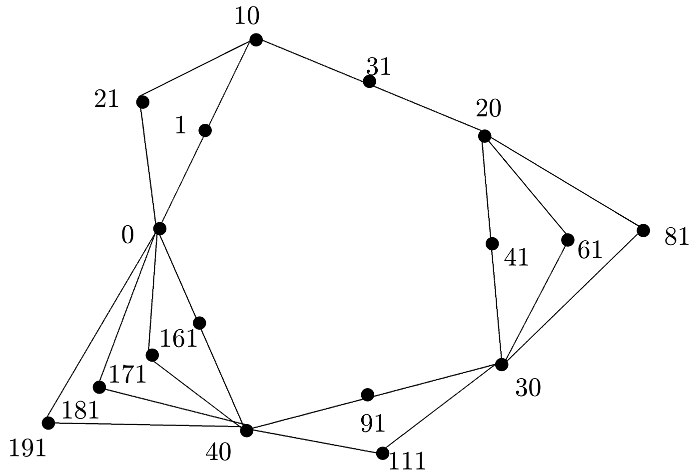
$$\phi(u_{i,i+1}^{(k)}) = \begin{cases} 2N(k-1)+1 & \text{if } i = 1 \\ N(2k+i-1) + 2N(m_1 + m_2 + \cdots + m_{i-1} - i) + 1 & \text{if } i = 2, 3, \ldots n-1 \end{cases}$$

For $k = 1, 2, 3, \ldots, m_n = n-1$

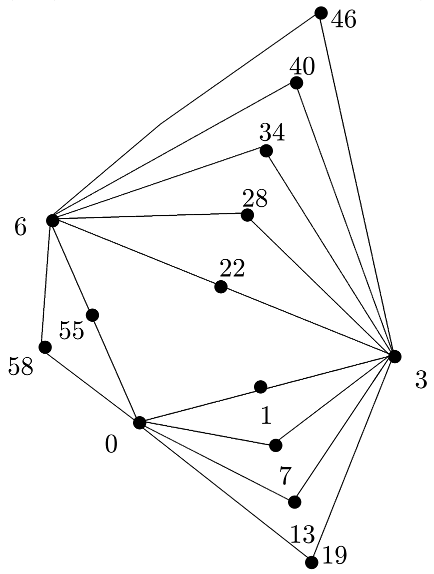
$\phi(v_{n,1}^{(k)}) = N[M - m_n - (n-k)] + 1.$

It is clear from the above labelling that the function ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$
Hence H is $(1, N)$-Arithmetic Labelling for every positive integer $N > 1$.
Clearly ϕ defines a $(1, N)$-Arithmetic Labelling of arbitrary supersubdivision of cycle C_n.

Example 2.12. $(1, 10)$-*Arithmetic Labelling of* $ASS(C_5)$



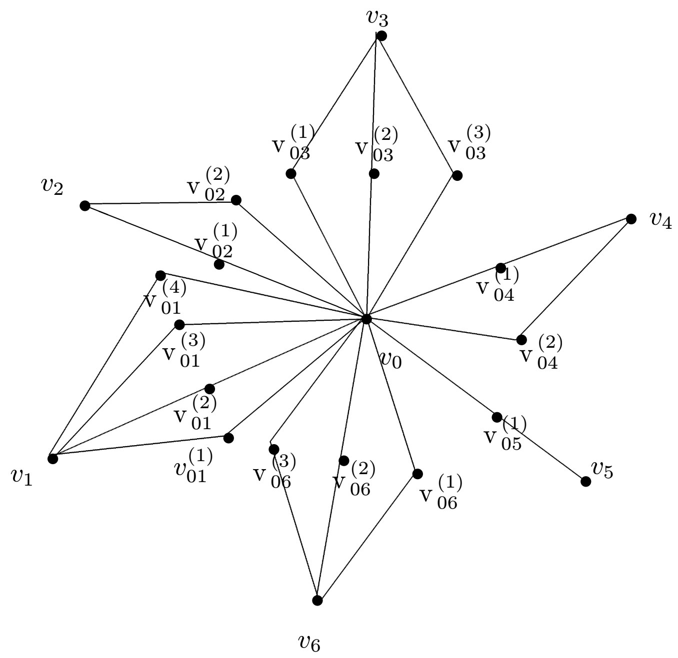
Example 2.13. $(1, 3)$-*Arithmetic Labelling of* $ASS(C_3)$



Theorem 2.14. *Arbitrary supersubdivision of any star is* $(1,N)$ *-Arithmetic Labelling for every positive integer* $N>1$ *.*

Proof. Let S_n be a star with vertices $v_0, v_1, v_2, ...v_n$ and let e_i denote the edge v_0v_i of S_n for $1 \le i \le n$. Let H be an arbitrary supersubdivision of S_n. That is for $1 \le i \le n$ each edge e_i of S_n is replaced by a complete bipartite graph K_{2,m_i} with m_i is any positive integer for $1 \le i \le n-1$ and $m_n = (n-1)$. It is clear that H has $M = 2(m_1 + m_2 + ... + m_n)$ *edges.*
The vertex set and edge set of H are given below:

$V(H) = \{v_0, v_1, v_2 \ldots, v_n, v_{01}^{(1)}, v_{01}^{(2)} \ldots, v_{01}^{(m_1)}, v_{02}^{(1)}, v_{02}^{(2)}, \ldots, v_{02}^{(m_2)}, \ldots, v_{0n}^{(1)}, v_{0n}^{(2)}, \ldots, v_{0n}^{(m_n)}\}$



An arbitrary supersubdivision of S_6

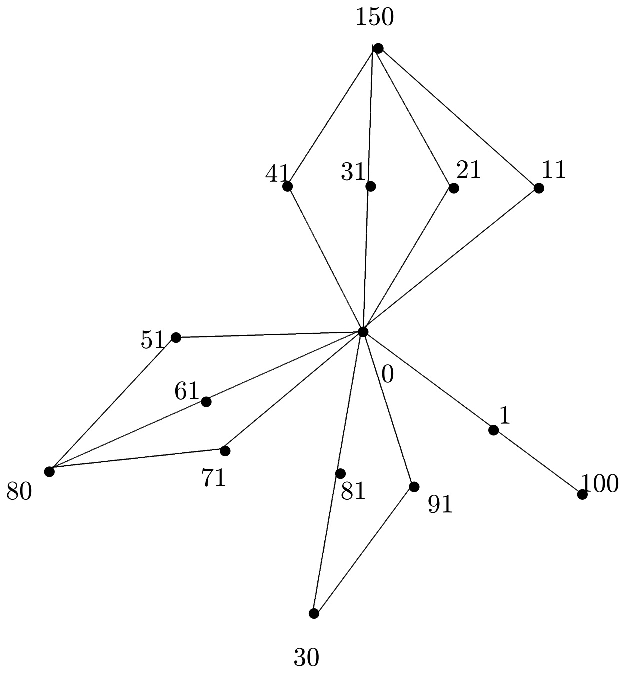
Define $\phi : V(H) \to \{0, 1, 2, \ldots 2\sum_{i=1}^{n} m_i\}$ as follows
$\phi(v_0) = 0$
For $k = 1, 2, 3, \ldots, m_i$

$$\phi(v_{0i}^{(k)}) = \begin{cases} N(k-1)+1 & \text{if } i = 1 \\ N(k-1)+1+N(m_1+m_2+\cdots+m_{i-1}) & \text{if } i = 2, 3, \ldots n \end{cases}$$

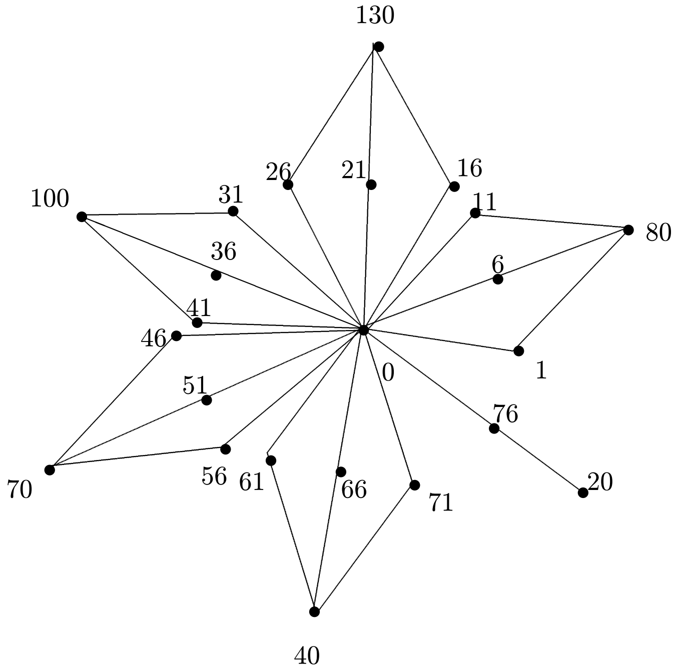
$$\phi(v_i) = \begin{cases} N(m_n-1)+N(m_1+m_2+\cdots+m_{n-1})+N & \text{if } i = 1 \\ N(m_1+m_i)+N[M-(2m_1+2m_2+\cdots+2m_i)] & \text{if } i = 2, 3, \ldots n \end{cases}$$

It is clear from the above labelling that the function ϕ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$
Hence H is $(1,N)$-Arithmetic Labelling for every positive integer $N > 1$.
Clearly ϕ defines a $(1,N)$-Arithmetic Labelling of arbitrary supersubdivision of star S_n.

Example 2.15. $(1,10)$ *-Arithmetic Labelling of* $ASS(S_4)$



Example 2.16. $(1, 5)$ *-Arithmetic Labelling of* $ASS(S_6)$



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