

# Multidimensional coincidence point results for generalized $(\psi, \theta, \varphi)$ -contraction on partially ordered metric spaces

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**Abstract.** The main objective of this research article is to establish some coincidence point theorem for  $g$ -non-decreasing mappings under generalized  $(\psi, \theta, \varphi)$ -contraction on a partially ordered metric space. Furthermore, we show how multidimensional results can be seen as a simple consequences of our uni-dimensional coincidence point theorem. Our results modify, improve, sharpen, enrich and generalize various known results.

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## 1. Introduction and Preliminaries

Roldan et al. [14] introduced the notion of multidimensional fixed/coincidence point which extended and generalized the fixed point results to higher dimensions. They used permutations of variables and distinguished between the first and the last variables.

In order to fix the framework needed to state our main results, we recall the following notions. For simplicity, we denote from now on  $X \times X \times \dots \times X$  ( $n$

times) by  $X^n$ , where  $n \in \mathbb{N}$  with  $n \geq 2$  and  $X$  is a non-empty set. Let  $\{A, B\}$  be a partition of the set  $\Lambda_n = \{1, 2, \dots, n\}$ , that is,  $A$  and  $B$  are nonempty subsets of  $\Lambda_n$  such that  $A \cup B = \Lambda_n$  and  $A \cap B = \emptyset$ . We will denote  $\Omega_{A,B} = \{\sigma : \Lambda_n \rightarrow \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$  and  $\Omega'_{A,B} = \{\sigma : \Lambda_n \rightarrow \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}$ . Henceforth, let  $\sigma_1, \sigma_2, \dots, \sigma_n$  be  $n$  mappings from  $\Lambda_n$  into itself and let  $\Upsilon$  be the  $n$ -tuple  $(\sigma_1, \sigma_2, \dots, \sigma_n)$ . Let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. For brevity, we denote  $g(x)$  by  $gx$  where  $x \in X$ .

A partial order  $\preceq$  on  $X$  can be extended to a partial order  $\sqsubseteq$  on  $X^n$  in the following way. If  $(X, \preceq)$  be a partially ordered space,  $x, y \in X$  and  $i \in \Lambda_n$ , we will use the following notations:

$$x \preceq_i y \Rightarrow \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases} \tag{1.1}$$

Consider on the product space  $X^n$  the following partial order: for  $Y = (y_1, y_2, \dots, y_i, \dots, y_n)$ ,  $V = (v_1, v_2, \dots, v_i, \dots, v_n) \in X^n$ ,

$$Y \sqsubseteq V \Leftrightarrow y_i \preceq_i v_i. \tag{1.2}$$

Notice that  $\sqsubseteq$  depends on  $A$  and  $B$ . We say that two points  $Y$  and  $V$  are comparable, if  $Y \sqsubseteq V$  or  $V \sqsubseteq Y$ . Obviously,  $(X^n, \sqsubseteq)$  is a partially ordered set.

**Definition 1.1** [9, 14, 15]. A point  $(x_1, x_2, \dots, x_n) \in X^n$  is called a  $\Upsilon$ -coincidence point of the mappings  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) = gx_i, \text{ for all } i \in \Lambda_n.$$

If  $g$  is the identity mapping on  $X$ , then  $(x_1, x_2, \dots, x_n) \in X^n$  is called a  $\Upsilon$ -fixed point of the mapping  $F$ .

It is clear that the previous definition extends the notions of coupled, tripled, and quadruple fixed points. In fact, if we represent a mapping  $\sigma : \Lambda_n \rightarrow \Lambda_n$  throughout its ordered image, that is,  $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ , then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when  $n = 2$ ,  $\sigma_1 = (1, 2)$  and  $\sigma_2 = (2, 1)$ ,

(ii) Berinde and Borcut's tripled fixed points are associated with  $n = 3$ ,  $\sigma_1 = (1, 2, 3)$ ,  $\sigma_2 = (2, 1, 2)$  and  $\sigma_3 = (3, 2, 1)$ ,

(iii) Karapinar's quadruple fixed points are considered when  $n = 4$ ,  $\sigma_1 = (1, 2, 3, 4)$ ,  $\sigma_2 = (2, 3, 4, 1)$ ,  $\sigma_3 = (3, 4, 1, 2)$  and  $\sigma_4 = (4, 1, 2, 3)$ .

These cases consider  $A$  as the odd numbers in  $\{1, 2, \dots, n\}$  and  $B$  as its even numbers.

**Definition 1.2** [14]. Let  $(X, \preceq)$  be a partially ordered space. We say that  $F$  has the mixed  $(g, \preceq)$ -monotone property if  $F$  is  $g$ -monotone non-decreasing in arguments of  $A$  and  $g$ -monotone non-increasing in arguments of  $B$ , that is, for all  $x_1, x_2, \dots, x_n, y, z \in X$  and all  $i$ ,

$$gy \preceq gz \Rightarrow F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \preceq_i F(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n).$$

**Definition 1.3** [15, 17]. Let  $(X, d)$  be a metric space and define  $\Delta_n, \rho_n : X^n \times X^n \rightarrow [0, +\infty)$ , for  $Y = (y_1, y_2, \dots, y_n), V = (v_1, v_2, \dots, v_n) \in X^n$ , by

$$\Delta_n(Y, V) = \frac{1}{n} \sum_{i=1}^n d(y_i, v_i) \text{ and } \rho_n(Y, V) = \max_{1 \leq i \leq n} d(y_i, v_i).$$

Then  $\Delta_n$  and  $\rho_n$  are metric on  $X^n$  and  $(X, d)$  is complete if and only if  $(X^n, \Delta_n)$ . Similarly  $(X, d)$  is complete if and only if  $(X^n, \rho_n)$  are complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k, Y) &\rightarrow 0 \text{ (as } k \rightarrow \infty) \Leftrightarrow d(y_i^k, y_i) \rightarrow 0 \text{ (as } k \rightarrow \infty), \\ \text{and } \rho_n(Y^k, Y) &\rightarrow 0 \text{ (as } k \rightarrow \infty) \Leftrightarrow d(y_i^k, y_i) \rightarrow 0 \text{ (as } k \rightarrow \infty), \quad i \in \Lambda_n, \end{aligned}$$

where  $Y^k = (y_1^k, y_2^k, \dots, y_n^k)$  and  $Y = (y_1, y_2, \dots, y_n) \in X^n$ .

**Lemma 1.1** [15, 17, 18]. Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. Let  $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_n)$  be an  $n$ -tuple of mappings from  $\Lambda_n$  into itself verifying  $\sigma_i \in \Omega_{A,B}$  if  $i \in A$  and  $\sigma_i \in \Omega'_{A,B}$  if  $i \in B$ . Define  $F_\Upsilon, G : X^n \rightarrow X^n$ , for all  $y_1, y_2, \dots, y_n \in X$ , by

$$F_\Upsilon(y_1, y_2, \dots, y_n) = \begin{pmatrix} F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, \dots, y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)}, y_{\sigma_2(2)}, \dots, y_{\sigma_2(n)}), \\ \dots, F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, \dots, y_{\sigma_n(n)}) \end{pmatrix}, \quad (1.3)$$

and

$$G(y_1, y_2, \dots, y_n) = (gy_1, gy_2, \dots, gy_n). \quad (1.4)$$

- (1) If  $F$  has the mixed  $(g, \preceq)$ -monotone property, then  $F_\Upsilon$  is monotone  $(G, \sqsubseteq)$ -non-decreasing.
- (2) If  $F$  is  $d$ -continuous, then  $F_\Upsilon$  is  $\Delta_n$ -continuous and  $\rho_n$ -continuous.
- (3) If  $g$  is  $d$ -continuous, then  $G$  is  $\Delta_n$ -continuous and  $\rho_n$ -continuous.
- (4) A point  $(y_1, y_2, \dots, y_n) \in X^n$  is a  $\Upsilon$ -fixed point of  $F$  if and only if  $(y_1, y_2, \dots, y_n)$  is a fixed point of  $F_\Upsilon$ .
- (5) A point  $(y_1, y_2, \dots, y_n) \in X^n$  is a  $\Upsilon$ -coincidence point of  $F$  and  $g$  if and only if  $(y_1, y_2, \dots, y_n)$  is a coincidence point of  $F_\Upsilon$  and  $G$ .
- (6) If  $(X, d, \preceq)$  is regular, then  $(X^n, \Delta_n, \sqsubseteq)$  and  $(X^n, \rho_n, \sqsubseteq)$  are also regular.

The commutativity and compatibility of the mappings are defined as follows.

**Definition 1.4** [14]. We will say that two mappings  $T, g : X \rightarrow X$  are commuting if  $gTx = Tgx$  for all  $x \in X$ . We will say that  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  are commuting if  $gF(x_1, x_2, \dots, x_n) = F(gx_1, gx_2, \dots, gx_n)$  for all  $x_1, x_2, \dots, x_n \in X$ .

The following notion was introduced in order to avoid the necessity of commutativity.

**Definition 1.5** [1, 4, 8, 11]. Let  $(X, d, \preceq)$  be a partially ordered metric space. Two mappings  $T, g : X \rightarrow X$  are said to be  $O$ -compatible if

$$\lim_{n \rightarrow \infty} d(gTx_n, Tgx_n) = 0,$$

provided that  $\{x_n\}$  is a sequence in  $X$  such that  $\{gx_n\}$  is  $\preceq$ -monotone, that is, it is either non-increasing or non-decreasing with respect to  $\preceq$  and

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} gx_n \in X.$$

The natural extension to an arbitrary number of variables is the following one.

**Definition 1.6** [1]. Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. Let  $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_n)$  be an  $n$ -tuple of mappings  $\Lambda_n$  into itself verifying  $\sigma_i \in \Omega_{A,B}$  if  $i \in A$  and  $\sigma_i \in \Omega'_{A,B}$  if  $i \in B$ . We will say that  $(F, g)$  is a  $(O, \Upsilon)$ -compatible pair if, for all  $i \in \Lambda_n$ ,

$$\lim_{m \rightarrow \infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \dots, gx_m^{\sigma_i(n)})) = 0,$$

whenever  $\{x_m^1\}, \{x_m^2\}, \dots, \{x_m^n\}$  are sequences in  $X$  such that  $\{gx_m^1\}, \{gx_m^2\}, \dots, \{gx_m^n\}$  are  $\preceq$ -monotone and

$$\lim_{m \rightarrow \infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)}) = \lim_{m \rightarrow \infty} gx_m^i \in X, \text{ for all } i \in \Lambda_n.$$

**Lemma 1.2** [1]. If  $F$  and  $g$  are  $(O, \Upsilon)$ -compatible, then  $F_\Upsilon$  and  $G$  are  $O$ -compatible.

The following definitions are usual in the field of fixed point theory.

**Definition 1.7** [1]. A partially ordered metric space  $(X, d, \preceq)$  is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence  $\{x_n\} \subseteq X$  such that  $\{x_n\} \rightarrow x$  and  $x_n \preceq x_{n+1}$  (respectively,  $x_n \succeq x_{n+1}$ ) for all  $n \geq 0$ , we have that  $x_n \preceq x$  (respectively,  $x_n \succeq x$ ) for all  $n \geq 0$ .  $(X, d, \preceq)$  is said to be regular if it is both non-decreasing-regular and non-increasing-regular.

**Definition 1.8** [5]. Let  $(X, \preceq)$  be a partially ordered set and let  $T, g : X \rightarrow X$  be two mappings. We say that  $T$  is  $(g, \preceq)$ -non-decreasing if  $Tx \preceq Ty$  for all  $x, y \in X$  such that  $gx \preceq gy$ . If  $g$  is the identity mapping on  $X$ , we say that  $T$  is  $\preceq$ -non-decreasing.

**Remark 1.2** [5]. If  $T$  is  $(g, \preceq)$ -non-decreasing and  $gx = gy$ , then  $Tx = Ty$ . It follows that

$$gx = gy \Rightarrow \left\{ \begin{array}{l} gx \preceq gy, \\ gy \preceq gx \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} Tx \preceq Ty, \\ Ty \preceq Tx \end{array} \right\} \Rightarrow Tx = Ty.$$

**Definition 1.9** [16]. An altering distance function is a function  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  which satisfied the following conditions:

- (i<sub>ψ</sub>) ψ is continuous and non-decreasing,
- (ii<sub>ψ</sub>) ψ(t) = 0 if and only if t = 0.

In this research paper, we obtain some coincidence point theorem for *g*-non-decreasing mappings under generalized (ψ, θ, φ)-contraction on a partially ordered metric space. Furthermore, we show how multidimensional results can be seen as simple consequences of our unidimensional coincidence point theorem. We modify, improve, sharpen, enrich and generalize the results of Alotaibi and Alsulami [2], Alsulami [3], Harjani et al. [6], Harjani and Sadarangani [7], Luong and Thuan [10], Nieto and Rodriguez-Lopez [12], Razani and Parvaneh [13] and many other famous results in the literature.

**2. Main results**

**Theorem 2.1.** Let (X, d, ≤) be a partially ordered metric space and let T, g : X → X be two mappings such that the following properties are fulfilled:

- (i) T(X) ⊆ g(X),
- (ii) T is (g, ≤)-non-decreasing,
- (iii) there exists x<sub>0</sub> ∈ X such that gx<sub>0</sub> ≤ Tx<sub>0</sub>,
- (iv) there exist an altering distance function ψ, an upper semi-continuous function θ : [0, +∞) → [0, +∞) and a lower semi-continuous function φ : [0, +∞) → [0, +∞) such that

$$\psi(d(Tx, Ty)) \leq \theta(d(gx, gy)) - \varphi(d(gx, gy)),$$

for all x, y ∈ X such that gx ≤ gy, where θ(0) = φ(0) = 0 and ψ(t) - θ(t) + φ(t) > 0 for all t > 0. Also assume that, at least, one of the following conditions holds.

- (a) (X, d) is complete, T and g are continuous and the pair (T, g) is O-compatible,
- (b) (X, d) is complete, T and g are continuous and commuting,
- (c) (g(X), d) is complete and (X, d, ≤) is non-decreasing-regular,
- (d) (X, d) is complete, g(X) is closed and (X, d, ≤) is non-decreasing-regular,
- (e) (X, d) is complete, g is continuous, the pair (T, g) is O-compatible and (X, d, ≤) is non-decreasing-regular.

Then T and g have a coincidence point.

**Proof.** We divide the proof into four steps.

**Step 1.** We claim that there exists a sequence {x<sub>n</sub>}<sub>n≥0</sub> ⊆ X such that {gx<sub>n</sub>} is ≤-non-decreasing and gx<sub>n+1</sub> = Tx<sub>n</sub>, for all n ≥ 0. Starting from x<sub>0</sub> ∈ X given in (iii) and taking into account that Tx<sub>0</sub> ∈ T(X) ⊆ g(X), there exists x<sub>1</sub> ∈ X such that Tx<sub>0</sub> = gx<sub>1</sub>. Then gx<sub>0</sub> ≤ Tx<sub>0</sub> = gx<sub>1</sub>. Since T is (g, ≤)-non-decreasing, Tx<sub>0</sub> ≤ Tx<sub>1</sub>. Now Tx<sub>1</sub> ∈ T(X) ⊆ g(X), so there exists x<sub>2</sub> ∈ X such that Tx<sub>1</sub> = gx<sub>2</sub>. Then gx<sub>1</sub> = Tx<sub>0</sub> ≤ Tx<sub>1</sub> = gx<sub>2</sub>. Since T is (g, ≤)-non-decreasing, Tx<sub>1</sub> ≤ Tx<sub>2</sub>. Repeating this argument, there exists a sequence {x<sub>n</sub>}<sub>n≥0</sub> such that {gx<sub>n</sub>} is ≤-non-decreasing, gx<sub>n+1</sub> = Tx<sub>n</sub> ≤ Tx<sub>n+1</sub> = gx<sub>n+2</sub> and

$$gx_{n+1} = Tx_n, \text{ for all } n \geq 0.$$

**Step 2.** We claim that {d(gx<sub>n</sub>, gx<sub>n+1</sub>)} → 0.

Now, by contractive condition  $(iv)$  and  $(i_\psi)$ , we have

$$\begin{aligned} \psi(d(gx_{n+1}, gx_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \theta(d(gx_n, gx_{n+1})) - \varphi(d(gx_n, gx_{n+1})). \end{aligned} \tag{2.1}$$

But we have  $\psi(d(gx_n, gx_{n+1})) - \theta(d(gx_n, gx_{n+1})) + \varphi(d(gx_n, gx_{n+1})) > 0$ .  
Then

$$\frac{\psi(d(gx_{n+1}, gx_{n+2}))}{\psi(d(gx_n, gx_{n+1}))} \leq \frac{\theta(d(gx_n, gx_{n+1})) - \varphi(d(gx_n, gx_{n+1}))}{\psi(d(gx_n, gx_{n+1}))} < 1.$$

Thus

$$\psi(d(gx_{n+1}, gx_{n+2})) < \psi(d(gx_n, gx_{n+1})).$$

Since  $\psi$  is non-decreasing, therefore

$$d(gx_{n+1}, gx_{n+2}) < d(gx_n, gx_{n+1}).$$

This shows that the sequence  $\{\delta_n\}_{n=0}^\infty$  defined by  $\delta_n = d(gx_n, gx_{n+1})$ , is a decreasing sequence of positive numbers. Then there exists  $\delta \geq 0$  such that

$$\lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} d(gx_n, gx_{n+1}) = \delta. \tag{2.2}$$

We shall prove that  $\delta = 0$ . Suppose to the contrary that  $\delta > 0$ . Taking  $n \rightarrow \infty$  in (2.1), by using the property of  $\psi, \theta, \varphi$  and (2.2), we obtain

$$\psi(\delta) \leq \theta(\delta) - \varphi(\delta),$$

so

$$\psi(\delta) - \theta(\delta) + \varphi(\delta) \leq 0,$$

which is a contradiction. Thus, by (2.2), we get

$$\delta = \lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} d(gx_n, gx_{n+1}) = 0. \tag{2.3}$$

**Step 3.** We claim that  $\{gx_n\}_{n \geq 0}$  is a Cauchy sequence in  $X$ . Suppose that  $\{gx_n\}_{n \geq 0}$  is not a Cauchy sequence. Then there exists an  $\varepsilon > 0$  for which we can find two sequences of positive integers  $\{m(k)\}$  and  $\{n(k)\}$  such that for all positive integers  $k$ , and

$$d(gx_{n(k)}, gx_{m(k)}) \geq \varepsilon \text{ for } n(k) > m(k) > k.$$

Assuming that  $n(k)$  is the smallest such positive integer, we get

$$d(gx_{n(k)-1}, gx_{m(k)}) < \varepsilon.$$

Now, by triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq d(gx_{n(k)}, gx_{m(k)}) \\ &\leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) \\ &< d(gx_{n(k)}, gx_{n(k)-1}) + \varepsilon, \end{aligned}$$

Letting  $k \rightarrow \infty$  in the above inequality, by using (2.3), we have

$$\lim_{k \rightarrow \infty} d(gx_{n(k)}, gx_{m(k)}) = \varepsilon. \tag{2.4}$$

By the triangle inequality, we have

$$d(gx_{n(k)+1}, gx_{m(k)+1}) \leq d(gx_{n(k)+1}, gx_{n(k)}) + d(gx_{n(k)}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)+1}).$$

Letting  $k \rightarrow \infty$  in the above inequalities, using (2.3) and (2.4), we have

$$\lim_{k \rightarrow \infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \varepsilon. \tag{2.5}$$

As  $n(k) > m(k)$ ,  $gx_{n(k)} \succeq gx_{m(k)}$ , by using contractive condition (iv), we have

$$\begin{aligned} \psi(d(gx_{n(k)+1}, gx_{m(k)+1})) &= \psi(d(Tx_{n(k)}, Tx_{m(k)})) \\ &\leq \theta(d(gx_{n(k)}, gx_{m(k)})) - \varphi(d(gx_{n(k)}, gx_{m(k)})). \end{aligned}$$

Letting  $k \rightarrow \infty$  in the above inequality, by using the property of  $\psi$ ,  $\theta$ ,  $\varphi$  and (2.4), (2.5), we have

$$\psi(\varepsilon) \leq \theta(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction due to  $\varepsilon > 0$ . This shows that  $\{gx_n\}_{n \geq 0}$  is a Cauchy sequence in  $X$ .

**Step 4.** We claim that  $T$  and  $g$  have a coincidence point distinguishing between cases (a) – (e).

Suppose now that (a) holds, that is,  $(X, d)$  is complete,  $T$  and  $g$  are continuous and the pair  $(T, g)$  is  $O$ -compatible. Since  $(X, d)$  is complete, therefore there exists  $z \in X$  such that  $\{gx_n\} \rightarrow z$ . Now  $Tx_n = gx_{n+1}$  for all  $n \geq 0$ , we also have that  $\{Tx_n\} \rightarrow z$ . As  $T$  and  $g$  are continuous, then  $\{Tgx_n\} \rightarrow Tz$  and  $\{ggx_n\} \rightarrow gz$ . Taking into account that the pair  $(T, g)$  is  $O$ -compatible, we deduce that  $\lim_{n \rightarrow \infty} d(gTx_n, Tgx_n) = 0$ . In such a case, we conclude that  $d(gz, Tz) = \lim_{n \rightarrow \infty} d(ggx_{n+1}, Tgx_n) = \lim_{n \rightarrow \infty} d(gTx_n, Tgx_n) = 0$ , that is,  $z$  is a coincidence point of  $T$  and  $g$ .

Suppose now that (b) holds, that is,  $(X, d)$  is complete,  $T$  and  $g$  are continuous and commuting. It is obvious because (b) implies (a).

Suppose now that (c) holds, that is,  $(g(X), d)$  is complete and  $(X, d, \preceq)$  is non-decreasing-regular. As  $\{gx_n\}_{n \geq 0}$  is a Cauchy sequence in the complete space  $(g(X), d)$ , so there exist  $y \in g(X)$  such that  $\{gx_n\} \rightarrow y$ . Let  $z \in X$  be any point such that  $y = gz$ . In this case  $\{gx_n\} \rightarrow gz$ . Indeed, as  $(X, d, \preceq)$  is non-decreasing-regular and  $\{gx_n\}$  is  $\preceq$ -non-decreasing and converging to  $gz$ , we deduce that  $gx_n \preceq gz$  for all  $n \geq 0$ . Applying the contractive condition (iv),

$$\psi(d(gx_{n+1}, Tz)) = \psi(d(Tx_n, Tz)) \leq \theta(d(gx_n, gz)) - \varphi(d(gx_n, gz)).$$

On taking  $n \rightarrow \infty$  in the above inequality, we get  $d(gz, Tz) = 0$ , that is,  $z$  is a coincidence point of  $T$  and  $g$ .

Suppose now that (d) holds, that is,  $(X, d)$  is complete,  $g(X)$  is closed and  $(X, d, \preceq)$  is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then,  $(g(X), d)$  is complete and  $(X, d, \preceq)$  is non-decreasing-regular. Thus (c) is applicable.

Suppose now that (e) holds, that is,  $(X, d)$  is complete,  $g$  is continuous, the pair  $(T, g)$  is  $O$ -compatible and  $(X, d, \preceq)$  is non-decreasing-regular. As  $(X, d)$  is complete, so there exists  $z \in X$  such that  $\{gx_n\} \rightarrow z$ . Since  $Tx_n = gx_{n+1}$  for all  $n \geq 0$ , we also have that  $\{Tx_n\} \rightarrow z$ . As  $g$  is continuous, then  $\{ggx_n\} \rightarrow gz$ . Furthermore, since the pair  $(T, g)$  is  $O$ -compatible, we have  $\lim_{n \rightarrow \infty} d(gTx_{n+1}, Tgx_n) = \lim_{n \rightarrow \infty} d(gTx_n, Tgx_n) = 0$ . As  $\{ggx_n\} \rightarrow gz$ , the previous property means that  $\{Tgx_n\} \rightarrow gz$ .

Indeed, as  $(X, d, \preceq)$  is non-decreasing-regular and  $\{gx_n\}$  is  $\preceq$ -non-decreasing and converging to  $z$ , we deduce that  $gx_n \preceq z$  for all  $n \geq 0$ . Applying the contractive condition (iv), we get

$$\psi(d(Tgx_n, Tz)) \leq \theta(d(ggx_n, gz)) - \varphi(d(ggx_n, gz)).$$

Taking  $n \rightarrow \infty$  in the above inequality, we get  $d(gz, Tz) = 0$ , that is,  $z$  is a coincidence point of  $T$  and  $g$ .

If we take  $\psi(t) = t$  and  $\varphi(t) = 0$  for all  $t \geq 0$  in Theorem 2.1, we obtain the following corollary.

**Corollary 2.2.** Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $T, g : X \rightarrow X$  be two mappings satisfying (i) – (iii) of Theorem 2.1 and there exists an upper semi-continuous function  $\theta : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$d(Tx, Ty) \leq \theta(d(gx, gy)),$$

for all  $x, y \in X$  such that  $gx \preceq gy$ , where  $\theta(0) = 0$  and  $t - \theta(t) > 0$  for all  $t > 0$ . Also assume that, at least, one of the conditions (a) – (e) of Theorem 2.1 holds. Then  $T$  and  $g$  have a coincidence point.

If we take  $\varphi(t) = 0$  and  $\theta(t) = k\psi(t)$  with  $0 \leq k < 1$ , for all  $t \geq 0$  in Theorem 2.1, we have the following corollary.

**Corollary 2.3.** Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $T, g : X \rightarrow X$  be two mappings satisfying (i) – (iii) of Theorem 2.1 and there exists an altering distance function  $\psi$  such that

$$\psi(d(Tx, Ty)) \leq k\psi(d(gx, gy)),$$

for all  $x, y \in X$  such that  $gx \preceq gy$ . Also assume that, at least, one of the conditions (a) – (e) of Theorem 2.1 holds. Then  $T$  and  $g$  have a coincidence point.

If we take  $\psi(t) = \theta(t)$  for all  $t \geq 0$  in Theorem 2.1, we obtain the following corollary.

**Corollary 2.4.** Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $T, g : X \rightarrow X$  be two mappings satisfying (i) – (iii) of Theorem 2.1 and there



exist an altering distance function  $\psi$  and a lower semi-continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$\psi(d(Tx, Ty)) \leq \psi(d(gx, gy)) - \varphi(d(gx, gy)),$$

for all  $x, y \in X$  such that  $gx \preceq gy$ , where  $\varphi(0) = 0$ . Also assume that, at least, one of the conditions (a) – (e) of Theorem 2.1 holds. Then  $T$  and  $g$  have a coincidence point.

If we take  $\psi(t) = \theta(t) = t$  for all  $t \geq 0$  in Theorem 2.1, we get the following corollary.

**Corollary 2.5.** Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $T, g : X \rightarrow X$  be two mappings satisfying (i) – (iii) of Theorem 2.1 and there exists a lower semi-continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$d(Tx, Ty) \leq d(gx, gy) - \varphi(d(gx, gy)),$$

for all  $x, y \in X$  such that  $gx \preceq gy$ , where  $\varphi(0) = 0$ . Also assume that, at least, one of the conditions (a) – (e) of Theorem 2.1 holds. Then  $T$  and  $g$  have a coincidence point.

If we take  $\psi(t) = \theta(t) = t$  and  $\varphi(t) = (1 - k)t$  with  $k < 1$  for all  $t \geq 0$  in Theorem 2.1, we get the following corollary.

**Corollary 2.6.** Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $T, g : X \rightarrow X$  be two mappings satisfying (i) – (iii) of Theorem 2.1 and

$$d(Tx, Ty) \leq kd(gx, gy),$$

for all  $x, y \in X$  such that  $gx \preceq gy$ , where  $k < 1$ . Also assume that, at least, one of the conditions (a) – (e) of Theorem 2.1 holds. Then  $T$  and  $g$  have a coincidence point.

**Example 2.1.** Let  $X = \mathbb{R}$  be a metric space with the metric  $d : X \times X \rightarrow [0, +\infty)$  defined by  $d(x, y) = |x - y|$ , for all  $x, y \in X$ , with the natural ordering of real numbers  $\leq$ . Let  $T, g : X \times X \rightarrow X$  be defined as

$$Tx = \frac{x^2}{3} \text{ and } gx = x^2 \text{ for all } x \in X.$$

Clearly,  $T$  and  $g$  satisfied the contractive condition of Theorem 2.1 with  $\psi(t) = \theta(t) = t$  and  $\varphi(t) = \frac{2t}{3}$  for  $t \geq 0$ . In addition, all the other conditions of Theorem 2.1 are satisfied and  $z = 0$  is a coincidence point of  $T$  and  $g$ .

### 3. Multidimensional coincidence point results

Next we give an  $n$ -dimensional fixed point theorem for mixed monotone mappings. For brevity,  $(y_1, y_2, \dots, y_n)$ ,  $(v_1, v_2, \dots, v_n)$  and  $(y_0^1, y_0^2, \dots, y_0^n)$  will be denoted by  $Y, V$  and  $Y_0$  respectively.

**Lemma 3.1.** Let  $(X, d, \preceq)$  be a partially ordered metric space and let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. Then

(1) If there exist  $y_0^1, y_0^2, \dots, y_0^n \in X$  verifying  $gy_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, \dots, y_0^{\sigma_i(n)})$ , for all  $i \in \Lambda_n$ , then there exists  $Y_0 \in X^n$  such that  $G(Y_0) \subseteq F_{\Upsilon}(Y_0)$ .

(2) If there exist an altering distance function  $\psi$ , an upper semi-continuous function  $\theta : [0, +\infty) \rightarrow [0, +\infty)$  and a lower semi-continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$\begin{aligned} & \psi(d(F(y_1, y_2, \dots, y_n), F(v_1, v_2, \dots, v_n))) \\ & \leq \theta\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right) - \varphi\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right), \end{aligned} \tag{3.1}$$

for which  $y_i, v_i \in X$  such that  $gy_i \preceq_i gv_i$  for all  $i \in \Lambda_n$ , where  $\theta(0) = \varphi(0) = 0$  and  $\psi(t) - \theta(t) + \varphi(t) > 0$  for all  $t > 0$ , then

$$\psi(\rho_n(F_{\Upsilon}(Y), F_{\Upsilon}(V))) \leq \theta(\rho_n(G(Y), G(V))) - \varphi(\rho_n(G(Y), G(V))), \tag{3.2}$$

for all  $Y, V \in X^n$  with  $G(Y) \subseteq G(V)$ .

**Proof.** (1) is obvious.

(2) Suppose that  $G(Y) \subseteq G(V)$  for  $Y, V \in X^n$ . For fixed  $i \in A$ , we have  $gy_{\sigma_i(t)} \preceq_t gv_{\sigma_i(t)}$  for  $t \in \Lambda_n$ . From (3.1), we have

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}), F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, \dots, v_{\sigma_i(n)}))) \\ & \leq \theta\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right) - \varphi\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right), \end{aligned} \tag{3.3}$$

for all  $i \in A$ . Similarly, for fixed  $i \in B$ , we have  $gy_{\sigma_i(t)} \succeq_t gv_{\sigma_i(t)}$  for  $t \in \Lambda_n$ . It follows from (3.1) that

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}), F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, \dots, v_{\sigma_i(n)}))) \\ & \leq \psi(d(F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, \dots, v_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}))) \\ & \leq \theta\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right) - \varphi\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right), \end{aligned} \tag{3.4}$$

for all  $i \in B$ . By (1.2), (1.3), (1.4), (3.3), (3.4) and  $\psi$  is non-decreasing, we have

$$\psi(\rho_n(F_{\Upsilon}(Y), F_{\Upsilon}(V))) \leq \theta(\rho_n(G(Y), G(V))) - \varphi(\rho_n(G(Y), G(V))),$$

for all  $Y, V \in X^n$  with  $G(Y) \subseteq G(V)$ .

**Theorem 3.1.** Let  $(X, \preceq)$  be a partially ordered set and suppose that there is a metric  $d$  on  $X$  such that  $(X, d)$  is a complete metric space. Let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings and  $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_n)$  be an  $n$ -tuple of mappings from  $\Lambda_n$  into itself verifying  $\sigma_i \in \Omega_{A,B}$  if  $i \in A$  and  $\sigma_i \in \Omega_{A,B}$  if  $i \in B$ . Suppose that the following properties are fulfilled:

(i)  $F(X^n) \subseteq g(X)$ ,

- (ii)  $F$  has the mixed  $g$ -monotone property,
- (iii) there exist  $y_0^1, y_0^2, \dots, y_0^n \in X$  verifying  $gy_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, \dots, y_0^{\sigma_i(n)})$ , for all  $i \in \Lambda_n$ ,
- (iv) there exist an altering distance function  $\psi$ , an upper semi-continuous function  $\theta : [0, +\infty) \rightarrow [0, +\infty)$  and a lower semi-continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying (3.1) for which  $y_i, v_i \in X$  such that  $gy_i \preceq_i gv_i$  for all  $i \in \Lambda_n$ , where  $\theta(0) = \varphi(0) = 0$  and  $\psi(t) - \theta(t) + \varphi(t) > 0$  for all  $t > 0$ . Also assume that at least one of the following conditions holds,
  - (a)  $(X, d)$  is complete,  $F$  and  $g$  are continuous and the pair  $(F, g)$  is  $(O, \Upsilon)$ -compatible,
  - (b)  $(X, d)$  is complete,  $F$  and  $g$  are continuous and commuting,
  - (c)  $(g(X), d)$  is complete and  $(X, d, \preceq)$  is non-decreasing-regular,
  - (d)  $(X, d)$  is complete,  $g(X)$  is closed and  $(X, d, \preceq)$  is regular,
  - (e)  $(X, d)$  is complete,  $g$  is continuous, the pair  $(F, g)$  is  $(O, \Upsilon)$ -compatible and  $(X, d, \preceq)$  is non-decreasing-regular.

Then  $F$  and  $g$  have a  $\Upsilon$ -coincidence point.

**Proof.** It is only necessary to apply Theorem 2.1 to the mappings  $T = F_{\Upsilon}$  and  $g = G$  in the ordered metric space  $(X^n, \rho_n, \sqsubseteq)$  taking into account all items of Lemma 1.1, Lemma 1.2 and Lemma 3.1.

**Theorem 3.2.** Let  $(X, \preceq)$  be a partially ordered set and suppose that there is a metric  $d$  on  $X$  such that  $(X, d)$  is a complete metric space. Let  $F : X^n \rightarrow X$  and  $g : X \rightarrow X$  be two mappings and  $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_n)$  be an  $n$ -tuple of mappings from  $\Lambda_n$  into itself verifying  $\sigma_i \in \Omega_{A,B}$  if  $i \in A$  and  $\sigma_i \in \Omega_{A,B}$  if  $i \in B$ . Suppose that the following properties are fulfilled:

- (i)  $F(X^n) \subseteq g(X)$ ,
- (ii)  $F$  has the mixed  $g$ -monotone property,
- (iii) there exist  $y_0^1, y_0^2, \dots, y_0^n \in X$  verifying  $gy_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, \dots, y_0^{\sigma_i(n)})$ , for all  $i \in \Lambda_n$ ,
- (iv) for which there exist an altering distance function  $\psi$ , an upper semi-continuous function  $\theta : [0, +\infty) \rightarrow [0, +\infty)$  and a lower semi-continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$\begin{aligned} & \psi \left( \frac{1}{n} \sum_{i=1}^n d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}), F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, \dots, v_{\sigma_i(n)})) \right) \\ & \leq \theta \left( \frac{1}{n} \sum_{i=1}^n d(gy_i, gv_i) \right) - \varphi \left( \frac{1}{n} \sum_{i=1}^n d(gy_i, gv_i) \right), \end{aligned} \tag{3.5}$$

for which  $y_i, v_i \in X$  such that  $gy_i \preceq_i gv_i$  for all  $i \in \Lambda_n$ , where  $\theta(0) = \varphi(0) = 0$  and  $\psi(t) - \theta(t) + \varphi(t) > 0$  for all  $t > 0$ . Also assume that at least one of the following conditions holds,

- (a)  $(X, d)$  is complete,  $F$  and  $g$  are continuous and the pair  $(F, g)$  is  $(O, \Upsilon)$ -compatible,
- (b)  $(X, d)$  is complete,  $F$  and  $g$  are continuous and commuting,

- (c)  $(g(X), d)$  is complete and  $(X, d, \preceq)$  is non-decreasing-regular,  
 (d)  $(X, d)$  is complete,  $g(X)$  is closed and  $(X, d, \preceq)$  is regular,  
 (e)  $(X, d)$  is complete,  $g$  is continuous, the pair  $(F, g)$  is  $(O, \Upsilon)$ -compatible and  $(X, d, \preceq)$  is non-decreasing-regular.

Then  $F$  and  $g$  have a  $\Upsilon$ -coincidence point.

**Proof.** Notice that the contractive condition (3.5) means that

$$\psi(\Delta_n(F_{\Upsilon}(Y), F_{\Upsilon}(V))) \leq \theta(\Delta_n(GY, GV)) - \varphi(\Delta_n(GY, GV)),$$

for all  $Y, V \in X^n$  with  $Y \sqsubseteq V$ . Therefore it is only necessary to use Theorem 2.1 to the mappings  $T = F_{\Upsilon}$  and  $g = G$  in the ordered metric space  $(X^n, \Delta_n, \sqsubseteq)$  taking into account all items of Lemma 1.1 and Lemma 3.1.

In a similar way, we may state the results analog of Corollary 2.2, Corollary 2.3, Corollary 2.4, Corollary 2.5 and Corollary 2.6 for Theorem 3.1 and Theorem 3.2.

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