Some Generalizations of Weak Convergence for any Lacunary Sequence

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Abstract

The main aim of the paper is to study and describe some generalizations of weak convergence for any lacunary sequence $\varphi = (l_r)$ *and an ideal* $I \subset P(\mathbb{M})$. *This weak convergences are represented by the notations* $WT(I)$ *as the convergence and* $WT\varphi(I)$ *as convergence on a Banach space Y.*

Keywords— *Weak convergence, Statistical convergence, Lacunary statistical convergence, Ideal convergence.*

I. Introduction

F Statistical convergence was introduced to assign limit to non-converging sequences in the usual intent using asymptotic density of subsets of $\mathbb{M}$, the set of positive integers (Fast [1951] and Schonenberg [1951]).

Definition 1.1

A sequence y= y_l of numbers converge statistically to a number N given that, for every €>0 (Fast [1951]),

$$\lim_m \frac{1}{m} |\{L \leq m : |y_l - N| \geq €\}| = 0, \quad (1)$$

where $y = y_l$ be a sequence of elements of Y, Y represent a Banach space ,{·} represents the cardinality of the enclosed set. In this condition, it is written as $T - \lim_{l\to\infty} y_l = N$. During last decades, many real-time issues arising in analysis have been resolved using statistical convergence. But recently, the statistical convergence faces the active process of alteration ([19], [9] and [3]) in many fields such as trigonometric series [21]), summability theory [8], intuitionistic fuzzy normed spaces [13], probabilistic normed spaces [13], locally convex spaces [17] and Banach spaces [14]. Statistical convergence is generalized to lacunary statistical convergence using lacunary sequence [10]. With the help of a lacunary sequence, the sequence $\varphi = (l_r)$ of positive integers is increased such that l_0=0 and $h_r = l_r - l_{r-1} \to \infty$ $as\ r \to \infty$.The intervals determined by φ will be denoted by $h_r = (l_{r-1}, l_r]$ and the ratio l_r/l_{r-1} is denoted by g_r.

Definition 1.2

For a lacunary sequence $\varphi = (l_r)$, the sequence y= y_l of numbers statistically lacunary converges to a number L provided that, for every €>0 [10],

$$\rho_\varphi(\{l \epsilon I_r : |y_l - N| \geq €\}) = 0 \quad (2)$$

where $\rho_{\varphi}(l) = \lim_r \frac{1}{h_r}|l\epsilon I_r: l\epsilon L\}|$. In this case, we write $T_{\varphi} - \lim_{l\to\infty} y_l = N$.

Definition 1.3

A sequence y=(y_l) of numbers is said to be M_{ϕ} converges to K provided that, for each €>0 [10],

$$\lim_r \frac{1}{h_r}\sum_{l\epsilon I_r}|y_l - N| = 0 \quad (3)$$

Here, it is possible to write $M_{\phi} - \lim_{l\to\infty} y_l = N$

Statistical convergence is generalized to weak statistical convergence and given the Banach spaces with separable duals using weak statistical convergence [4]. Assume Y be a Banach space and (y_l) be a sequence in Y.

Definition 1.4

A sequence (y_l) converges weak statistically to y∈Y provided that, for each f∈Y*,(Y* is the continuous dual of Y), each €>0 [4],

$$\rho(\{l \le m: |f(y_l - y)| \ge €\}) = 0 \quad (4)$$

Here, it is possible to write $WT - \lim_{l\to\infty} y_l = y$. Although, weak statistical Cauchy sequences is introduced in a normed space and weak statistical convergence in n_p spaces [1]). Weak statistical convergence is generalized to lacunary weak statistical convergence as follows [18]).

Definition 1.5 A sequence (y_l) converges weak lacunary statistically to y∈Y provided that, for each f∈ Y*, each €>0 [18],

$$\rho(\{l \le m: |f(y_l - y)| \ge €\}) = 0 \quad (5)$$

Here, it is possible to write $WT_{\varphi} - \lim_{l\to\infty} y_l = y$.

Definition 1.6

A sequence y_l is weak and M_{ϕ} is convergent to y∈Y provided that, for each f∈ Y*, each €>0 [18],

$$\lim_r \frac{1}{h_r}\sum_{l\epsilon I_r}|f(y_l - y| = 0 \quad (6)$$

Then a brief description of I−convergence is given. I−convergence is defined as a natural generalization of statistical convergence [16]. For $\mathbb{M}$, the set of positive integers, assume $P(\mathbb{M})$ be the power set of $\mathbb{M}$.

Definition 1.7

A family of sets $I \subset P(\mathbb{M})$ is said as an ideal in $\mathbb{M}$ iff

(a) $\alpha \in I$;

(b) For each $A,B\in I$ we have $A\cup B\in I$;

(c) For $A\in I$ and $B\subseteq A$ we have $B\in I$.

Definition 1.8

A non-empty family of sets $F\subset P(\mathbb{M})$ is said to be a filter on $\mathbb{M}$ iff

(a) $\alpha \notin F$;

(b) For each $A,B\in F$ we have $A\cap B\in F$;

(c) For $A\in F$ and $B\supseteq A$ we have $B\in F$.

An ideal I is said to be non-trivial if $I\neq P(\mathbb{M})$. It soon implies that $I\subset P(\mathbb{M})$ is a non-trivial ideal provided that the class $F=F(I)=\{\mathbb{M}-A:A\in I\}$ is a filter on $\mathbb{M}$ which represents the filter correlated with the ideal I. A non-trivial ideal $I\subset P(\mathbb{M})$ denotes an admissible ideal in $\mathbb{M}$ if and only if it has all $\{\{x\}:x\in \mathbb{M}\}$ (called singletons).

Definition 1.9

A sequence y=y_l of numbers is said to be $I-$convergent to a number N provided that, for every €>0,

$$\{l \in \mathbb{M}: |y_l - N| \geq €\} \in I \quad (7)$$

Further, the extended idea of $I-$convergence to $I-$statistical convergence and $I-$lacunary statistical convergence as follows (Das *et al.* [2011]).

Definition 1.10 A sequence y= y_l of numbers converges $I-$statistically to L provided that, for every €>0 and each β>0,

$$\{m \in \mathbb{M}: \frac{1}{m}|l \leq m: |(y_l - N)| \geq €\}| \geq \beta\} \in I \quad (8)$$

Moreover, above definition is also generalized for lacunary sequences as follows.

Definition 1.11

Assume $\varphi=(l_r)$ be a lacunary sequence, then a sequence y= y_l of numbers is said to be $I-$lacunary statistically convergent to L or $S_\theta(I)-$convergent to L provided that, for every €>0 and each β >0,

$$\{r \in \mathbb{M}: \frac{1}{h_r}|l\epsilon I_r: |y_l - N| \geq €\}| \geq \beta\} \in I \quad (9)$$

Here, it is possible to write $y_l\rightarrow$N(T φ (I)).

Definition1.12

A sequence $y = y_l$ of numbers is M_ϕ (I) converges to N provided that, for every €>0,

$$\{r \in \mathbb{M}: \frac{1}{h_r}|l \epsilon I_r : \Sigma_{l \epsilon I_r} y_l - N \geq €\}| \in \mathrm{I} \tag{10}$$

Here, it is possible to write $y_l \to$N(M_ϕ (I)). Further, $I-$convergence found its own applications [4], [11] and [15].

II.WEAK I− LACUNARY STATISTICAL CONVERGENCE

Weak lacunary statistical convergence is defined and proved some relations among weak lacunary statistical convergence and weak statistical convergence (Nuray [2011]). Here, these notions are generalized and relevent connections are studied.

Definition 2.1

A sequence (y_l) converges weak statistically to y with respect to I in Y if for every €>0, every β >0 and each $f\in$ Y*, where Y* is the continuous dual of Y, then

$$\{m \in \mathbb{M}: \frac{1}{m}|l \leq m : |f(y_l - y)| \geq €\}| \geq \beta\} \in \mathrm{I} \tag{11}$$

Here, it is possible to write $WT(I) - \ lim x_k = x$. The set of all weak $I-$ statistically convergent sequences is denoted by WT(I). For I=I_{finite}, weak $I-$statistical convergence agrees with weak statistical convergence of (*Bhardwaj et al* [*2007*]).

Definition 2.2

A sequence (y_l) converges weak lacunary statistically to y with respect to I in X if for every €>0, every β >0 and each $f\in$ Y*, where Y* is the continuous dual of Y, then

$$\left\{r \in \mathbb{M}: \frac{1}{h_r}|l \epsilon I_r : |f(y_l - y)| \geq €\}| \geq \beta\right\} \in \mathrm{I}. \tag{12}$$

Here, it is possible to write $WT_\phi(I) - lim x_k = x$. The set of all weak $I-$ lacunary statistically convergent sequences is denote by $W\,T_\phi\,(I)$. For I=I_{finite}, weak $I-$lacunary statistical convergence agrees with weak lacunary statistical convergence of (Nuray [2011]).

Definition 2.3

A sequence (y_l) is weak M_ϕ is convergent to y with respect to I in Y if for every €>0 and each $f\in$ Y*,

$$\left\{r \in \mathbb{M}: \frac{1}{h_r}\Sigma_{l \epsilon I_r}|f(y_l - y)| \geq €\right\} \in \mathrm{I}. \tag{13}$$

Here, it is possible to write WM_ϕ $(I) - lim x_k = x$. The set of all weak M_ϕ (I) which is the convergent sequences is denoted by W M φ (I). For I=I_{finite}, weak M_ϕ (I) agrees with weak M_ϕ (Nuray [2011]).

III. WEAK I−LACUNARY STATISTICAL LIMIT AND CLUSTER POINT

The notion of statistical limit point and statistical cluster point for the sequences of numbers are introduced (Fridy [1993]). These notions to A as statistical limit point and A as statistical cluster point is generalised (Connor *et al.* [2000]) where A is a non-negative regular summability matrix. These notions is generalised to φ as statistical limit point and φ as statistical cluster point for sequences of numbers (Demirci [2002]). The notion of A as statistical limit point and A as statistical cluster point to define AI as statistical limit point and AI as statistical cluster point where $I\subset P(\mathbb{M})$ is an ideal (Gürdal *et al.* [2013]). In addition these concepts has been described in different spaces. Now, the notion of weak I−lacunary statistical limit and cluster point in Banach spaces is introduced. Let $\varphi = l_r$ be a lacunary sequence. For a Banach space Y, let y= y_l be a sequence in Y. Let $y_l(j)$ j be a subsequence of y and L= {l(j) :$j\in\mathbb{M}$}, then $y_l(j)$ is denoted by $(y)_l$. If for every β >0,

$$\left\{r\in\mathbb{M}:\frac{1}{h_r}|l(j)\in I_r : j\in\mathbb{M}|\geq\beta\right\}\in I \tag{14}$$

then $(y)_l$ is called φ(I) as thin subsequence. on the other hand (y)l is a φ(I) as nonthin subsequence of y iff,

$$\left\{r\in\mathbb{M}:\frac{1}{h_r}|l(j)\in I_r : j\in\mathbb{M}|\geq\beta\right\}\in I \tag{15}$$

Definition 3.1

Let Y be a Banach space, $\varphi=(l_r)$ be a lacunary sequence and $f\in Y^*$. An element y∈Y is a weak Tφ (I) which is limit point of a sequence $y=(y_l)$ in Y provided that there is a φ (I)-nonthin subsequence of (y_l) that is weak convergent to y. Let Λ(WTφ (I),y) represents the set of all WT_ϕ (I)-limit points of the sequence $y=(y_l)$.

Definition 3.2

Let Y be a Banach space, $\varphi=(l_r)$ be a lacunary sequence and $f\in Y^*$. An element $x\in X$ is a weak Tφ I which is a cluster point of a sequence $y=y_l$ in Y provided that for every €>0 and each $f\in Y^*$, then

$$\left\{r\in\mathbb{M}:\frac{1}{h_r}|l\epsilon I_r:|f(y_l-y)|<€\}|\geq\beta\right\}\notin I. \tag{15}$$

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