Regularity Concept of L-Fuzzy Bi-Ideals in Near-Ring

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Abstract:
In this paper, we apply the idea of near ring. We introduce the notion of L-fuzzy bi-ideal in near-ring. Also we give conditions for a near ring with unity to be strongly regular in terms of fuzzy set.

Keywords: Near-ring, bi-ideals, fuzzy bi-ideals, L-fuzzy bi-ideals.

1. Introduction


2. Preliminaries

For the sake of continuity we recall some basic definitions.

2.1 Definition:
A non empty set N with two binary operations ‘+’ and ‘.’ is called a near-ring if
(i) (N,+) is a group.
(ii) (N,.) is a semigroup
(iii) x.(y+z) = x.y + x.z for all x, y, z ∈ N

Example:
Let N = { p, q, r, s} be a set with two binary operations as follows:

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(N,+,.) is a near-ring.
2.2 Definition:
A fuzzy set $\mu_A$ in a near ring $N$ is called a fuzzy ideal of $N$ if it satisfies:

(i) $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}$
(ii) $\mu_A(y + x - y) \geq \mu_A(x)$
(iii) $\mu_A(xy) \geq \mu_A(y)$
(iv) $\mu_A((x + z)y - xy) \geq \mu_A(z)$ for all $x, y, z \in N$

2.3 Definition:
A fuzzy set $\mu_A$ in $N$ is a fuzzy bi ideal of $N$ if

(i) $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}$ for all $x, y \in N$
(ii) $\mu_A(x y z) \geq \min \{ \mu_A(x), \mu_A(y) \}$ for all $x, y, z \in N$

2.4 Definition:
A fuzzy set $\mu_A$ in $N$ is a fuzzy sub near-ring of $N$ if,

(i) $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}$
(ii) $\mu_A(x y) \geq \min \{ \mu_A(x), \mu_A(y) \}$ for all $x, y \in N$

2.5 Definition:
A fuzzy set $\mu_A$ in a set $N$ is a function $\mu_A : N \rightarrow [0, 1]$. Denote by $\text{Im}(\mu_A)$ the image set of $\mu_A$. For $t \in [0, 1]$ the set $\mu_A^{-t} = \{ x \in N / \mu_A(x) \geq t \}$ (rep. $\mu_A^{-t} = \{ x \in N / \mu_A(x) \leq t \}$) is called a upper (resp. lower) $t$-level cut of $\mu_A$.

3. MAIN RESULTS

3.1 Definition:
A fuzzy set $\mu_A$ in $N$ is a L-fuzzy sub near-ring of $N$ if for all $x, y \in N$,

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$
(ii) $\mu_A(x y) \geq \mu_A(x) \wedge \mu_A(y)$

3.2 Definition:
A fuzzy set $\mu_A$ in $N$ is a L-fuzzy bi-ideal of $N$ if for all $x, y, z \in N$,

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$
(ii) $\mu_A(x y z) \geq \mu_A(x) \wedge \mu_A(z)$

Example:
Consider the fuzzy subset $\mu_A$ of $R$ defined by,

$$\mu_A(x) = \begin{cases} 0.6 & \text{if } x \in \langle 2 \rangle \\ 0.2 & \text{otherwise} \end{cases}$$

Then $\mu_A$ is L-fuzzy bi-ideal of $N$.

3.3 Theorem:
A fuzzy set $\mu_A$ in $N$ is an L-fuzzy bi-ideal of $N$ iff $\mu_A^c$ is an L-fuzzy bi-ideal of $N$.

Proof:
If $\mu_A$ is an L-fuzzy bi-ideal of $N$.

To prove:
$\mu_A^c$ is an L-fuzzy bi-ideal of $N$. $\mu_A^c$ is a L-fuzzy bi-ideal of $N$. 


\[ \mu_A^C(x - y) = 1 - \mu_A(x - y) \]
\[ \geq 1 - (\mu_A(x) \wedge \mu_A(y)) \]
\[ \geq (1 - \mu_A(x)) \wedge (1 - \mu_A(y)) \]
\[ = \mu_A^C(x) \wedge \mu_A^C(y) \]

Therefore, \( \mu_A^C(x - y) \geq \mu_A(x) \wedge \mu_A^C(y) \) for all \( x, y \in N \)

\[ \mu_A^C(x \ y \ z) = 1 - \mu_A(x \ y \ z) \]
\[ \geq 1 - (\mu_A(x) \wedge \mu_A(z)) \]
\[ \geq (1 - \mu_A(x)) \wedge (1 - \mu_A(z)) \]
\[ = \mu_A^C(x) \wedge \mu_A^C(z) \]

Therefore, \( \mu_A^C(x \ y \ z) \geq \mu_A(x) \wedge \mu_A^C(z) \) for all \( x, y, z \in N \)

\( \mu_A^C \) is an \( L \)-fuzzy bi-ideal of \( N \).

Conversely, \( \mu_A^C \) is an \( L \)-fuzzy bi-ideal of \( N \), then clearly the condition (i) and (ii) of Definition 3.2 are valid.

### 3.4 Theorem:

A fuzzy set \( \mu_A \) in \( N \) is a \( L \)-fuzzy bi-ideal of \( N \) if and only if all the non-empty sets \( \mu_i^+ \) and \( \mu_i^- \) are bi-ideal of \( N \) for all \( t \in \text{Im}(\mu_A) \)

**Proof:**

Suppose that \( \mu_A \) is an \( L \)-fuzzy bi-ideal of \( N \).

For \( x, y \in \mu_i^+ \), we have

\[ \mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) \]
\[ \geq t \]

Therefore, \( x - y \in \mu_i^+ \)

Let \( x, z \in \mu_i^+ \) and \( y \in N \). Then

\[ \mu_A(x \ y \ z) \geq \mu_A(x) \wedge \mu_A(z) \]
\[ \geq t \]

And so \( x \ y \ z \in \mu_i^+ \)

Hence \( \mu_i^+ \) is an \( L \)-fuzzy bi-ideal of \( N \) for all \( t \in \text{Im}(\mu_A) \) also \( \mu_i^- \) are \( L \)-fuzzy bi-ideal of \( N \) for all \( t \in \text{Im}(\mu_A) \)

Conversely,

Suppose that \( \mu_i^+ \) and \( \mu_i^- \) are bi-ideals of \( N \) for all \( t \in \text{Im}(\mu_A) \)

Suppose that \( x, y \in N \)

\[ \mu_A(x - y) \leq \mu_A(x) \wedge \mu_A(y) \]

Choose \( r \), such that \( \mu_A(x - y) < r < \mu_A(x) \wedge \mu_A(y) \) Then we get \( x, y \in \mu_i^+ \) but \( x - y \notin \mu_i^+ \) a contradiction. Therefore

\[ \mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) \]

Similarly \( \mu_A(x \ y \ z) \geq \mu_A(x) \wedge \mu_A(z) \) for all \( x, y, z \in N \)

Hence \( \mu_A \) is \( L \)-fuzzy bi-ideal of \( N \).

### 3.5 Theorem:

A non-empty set \( B \) of \( N \) is a bi-ideal of \( N \) if and only if \( A = (\mathcal{X}_B, \mathcal{X}_B^C) \) is an \( L \)-fuzzy bi-ideal of \( N \).

**Proof:**

Straightforward
4. L-FUZZY BI-IDEALS AND REGULARITY

A near-ring $N$ is regular if for every $a \in N$ there is an $x \in N$ such that $a = axa$.

A near ring $N$ is strongly regular if for every $a \in N$ there is an $x \in N$ such that $a = xaa$.

4.1 Theorem:

Every L-fuzzy bi-ideal in a regular near ring is a L-fuzzy sub near-ring of $N$.

Proof:

Let $\mu_a$ be a L-fuzzy bi-ideal of $N$ and Let $a, b \in N$.

Since $N$ is regular, there exist $x \in N$ such that $a = axa$.

Then $\mu_a(ab) = \mu_a((axa)b)$

$= \mu_a(a(xa)b)$

$\geq \mu_a(a) \wedge \mu_a(b)$

$\therefore \mu_a(ab) \geq \mu_a(a) \wedge \mu_a(b)$

Thus $\mu_a$ is a L-fuzzy sub near-ring of $N$.

4.2 Lemma:

Let $N$ be a strongly regular near ring. If $a = xaa$ for some $a, x \in N$, then $a = axa$ and $ax = xaa$.

4.3 Theorem:

If $N$ is strongly regular, then for each $a \in N$ there is some $y \in N$ such that $a = a^2 ya^2$.

Proof:

Since $N$ is strongly regular for each $a \in N$ there is an $x \in N$ such that $a = xaa$.

Then by lemma 4.2

$a = axa$ and $ax = xaa$

$\therefore a = (ax) = a^2 x$ and hence,

$a = axa = (a^2 x)x(a^2 x)$

$= a^2 ya^2$.

4.4 Theorem:

Let $N$ be a strongly regular near ring. Then for every L-fuzzy B-ideal in $N$, we have $\mu_a(x) = \mu_a(x^2)$ for all $x \in N$.

Proof:

Let $\mu_a$ be a L-fuzzy Bi-ideal of $N$ and let $x \in N$ since $N$ is strongly regular, there exist $y \in N$ such that $x = x^2 yx^2$ then,

$\mu_a(x) = \mu_a(x^2 yx^2)$

$\geq \mu_a(x^2) \wedge \mu_a(x^2)$

$= \mu_a(x^2)$.

4.5 Theorem:

Let $N$ be a near-ring with identity 1 if every L-fuzzy Bi-ideal $\mu_a$ in $N$ satisfies $\mu_a(x) = \mu_a(x^2)$ for all $x \in N$ then $N$ is strongly regular.

Proof:

Suppose that every L-fuzzy Bi-ideal $\mu_a$ of $N$ satisfies $\mu_a(x) = \mu_a(x^2)$ for all $x \in N$.

Let $a \in N$ then $B = Na^2$ is a Bi-ideal of $N$. 

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\[ A = (\mathcal{X}_B, \mathcal{X}_B^C) \] is an L-fuzzy Bi-ideal of N (by Theorem 3.5)

Since \( a^2 \in B = Na^2, \mathcal{X}_B(a^2) = 1 \).

But by hypothesis \( \mathcal{X}_B(a) = \mathcal{X}_B(a^2) \)

\[ \therefore \mathcal{X}_B(a) = 1 \] and so \( a \in B = Na^2 \)

Thus \( a = xa^2 \) for some \( x \in N \).

Hence N is strongly regular.

**Reference:**


