ON H-SUPER MAGIC TOTAL LABELINGS OF DISJOINT UNION OF CONNECTED GRAPHS

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Abstract

Let $G = (V(G), E(G))$ be a simple graph of order p and size q. Let $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ be a bijective function and $H = (V(H), E(H))$ be a subgraph of G. The weight of H, $wt(H)$, is the sum of the labels of its vertices and edges. We say that G admits an H-covering, if every edge in $E(G)$ belongs to at least one subgraph H_i, $1 \leq i \leq k$ of G isomorphic to H. We say that f is an H-magic total labeling if G admits an H-covering and for each subgraph H' of G isomorphic to H, the H'weight $wt(H')$ equals a fixed constant. Such a labeling is called H-super magic if the smallest possible labels appear on the vertices.

In this paper we study H-super magic total labeling of disjoint union of connected graphs.

Key Words. H-covering, H-magic total labeling, H-super magic total labeling.
AMS Subject Classification Number. 05C78.

1 Introduction

All graphs in this paper are finite, undirected and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set respectively. A (p,q)-graph G is a graph such that $|V(G)| = p$ and $|E(G)| = q$. For graph theoretic terminology, we follow **[1, 2]**.

An edge-covering of G is a family of subgraphs $H_1, H_2, H_3, \ldots, H_k$ such that any edge of $E(G)$ belongs to at least one of the subgraphs H_j's, $1 \leq j \leq k$. If every subgraph H_j is isomorphic to a given graph H, then G admits an H-covering.

A labeling of a graph G is a mapping from the set of vertices, edges or both vertices and edges to the set of numbers, usually integers. Based on the domain we distinguish vertex labeling, edge labeling and total labeling. Many kinds of labelings have been studied and an excellent survery of graph labeling can be found in **[3]**.

Suppose G admits an H-covering. A bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p+q\}$ is called an H-magic labeling of G if for each sub graph H' isomorphic to H, there exists a positive integer c such that the H' weight,

$$wt(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = c$$

and c is called magic constant. In this case, G is called H-magic. When $f(V(G)) = \{1, 2, 3, \ldots, p\}$, we say that G is H-super magic.

The notion of H-magic labeling, which is a generalization of Kotzig and Rosa's [12] edge magic total labeling, was introduced by Gutiérrez and Lladó [5] in 2005.

On the other hand, Inayah et al. [6] introduced an (a, d)-H-antimagic total labeling of G in 2009,which is defined as a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}$ such that for all subgraphs H' isomorphic to H, the set of H'-weights

$$wt(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

constitutes an arithmetic progression $a, a+d, a+2d, \ldots, a+(n-1)d$ where a and d are some positive integers and n is the number of subgraphs isomorphic to H. In this case we say that G is (a, d)-H-antimagic. When $f(V(G)) = \{1, 2, 3, \ldots, p\}$, we say that f is a super (a, d)-H-antimagic total labeling and G is super (a, d)-H-antimagic.

In [5], they studied the H-super magic labeling of some classes of connected graphs namely,stars,paths,cycles and complete bipartite graphs. In 2007, Lldó and Morogas [13], studied C_h-supermagic labelings of some graphs for some h, namely, wheels, windmills, prisms and books. In 2009,Inayah et al.[6] studied some properties of (a, d)-H-antimagic total labeling for any graph and also discussed the (a, d)-C_h-antimagic total labelings of fans. In 2010, Ngurah, Salman and Susilowati [17] studied the cycle-super magic labeling of chain graphs, fans, triangular ladders, grids, books and the graphs obtained by joining a star $K_{1,n}$ with one isolated vertex.Maryati et al. [15] studied the H-super magic labeling of some graphs obtained from k isomorphic copies of conected graph H. In [9], Jeyanthi and Selvagopal showed that one point union of garland graph and linear garland admit H-super magic.Salman and Maryati [18] proved that a path-amalgamation of isomorphic graphs are H-super magic. Salman et al. [19] studied some cycle-(super)magic labelings of some complete bipartite graphs. In 2012. Roswitha and Baskoro [20] studied the H-super magic labeling for some classes of trees such as double star, a caterpillar, a firecracker and a banana tree. In 2013, Kojima [11] studied the C_4-super magic labeling of the cartesian product of paths and graphs. In[8], Inayah, Simanjuntak and Salman proved that there exists a super (a, d)-H-antimagic total labeling for shack- les of a connected graph. In 2015, Maryati et al. [16] studied the path- (super) magicness of a cycle with some pendants. In[3], David Laurence and Kathiresan studied some properties of super(a, d)-H-antimagic total labeling for any graph and also discussed the super(a, d)-P_h-antimagic total labelings of Stars. In[10], Kathiresan and David Laurence studied the super (a, d)-H- antimagic total covering of star related graphs.

In this paper we study H-super magicness of disjoint union of t copies of connected graph G.

2 Sum Set Partitions

The proof of our main results is based on the use of sum set partition. We recall in this section some useful facts on this concept.

Let $x < y$ be positive integers. Throughout the paper we denote by $[x, y]$ to mean $\{i \in N : x \leq i \leq y\}$. Given a set X of integers and a partition $P = \{X_1, X_2, .., X_m\}$ of X into m parts, we denote $\sum(P) = (\sum X_1, \sum X_2, \ldots, \sum X_m)$, the sum set partition of P where $\sum X_i = \sum_{x \in X_i} x$. We will always order the partition in such a way that the sequence of subset sums $\sum X_1 \leq \sum X_2 \leq \cdots \leq \sum X_k$ is non decreasing.

When all sets in P have the same cardinality then we say that P is an equipartition of X or m-equipartition or a m-balanced multisets of X.

We have the following lemma.

Lemma 2.1. *Let l, m and n be non negative integers. For $1 \leq i \leq m$, let $A_i = n(i-1)+1, n(i-1)+n]$ with $|A_i| = n$, $B_i = [mn + n(i-1)+1, mn+n(i-1)+n]$ with $|B_i| = n$ and for $0 \leq l \leq \left\lfloor \frac{n^2-3n}{4} \right\rfloor$, let $C_l = [2mn+1, 2mn+lm]$ with $|C_l| = lm$, and $D_l = [2mn+lm+1, 2mn+2lm]$ with $|D_l| = lm$. Then there exists a partition Y of $\bigcup_{i=1}^{m}[(A_i \cup B_i)\} \cup C_l \cup D_l$ such that $\sum Y$ is an arithmetic progression starting at $(l+n)[2m(l+n)+1]$ with common difference 0 and hence Y is m-balanced with all its subsets being $2(n+l)$- sets.*

Proof. For each $i \in [1, m]$, define the $2(n+l)$-sets, $Y_i = \{a_{ij}, b_{ij} : 1 \leq j \leq n\} \cup \{c_{ij}, d_{ij} : 1 \leq j \leq l\}$ such that

$$a_{ij} = n(i-1)+j,\ 1 \leq j \leq n$$
$$b_{ij} = 2mn - n(i-1) - j + 1,\ 1 \leq j \leq n$$
$$c_{ij} = 2mn + j,\ 1 \leq j \leq l$$
$$d_{ij} = 2m(n+l) - j + 1,\ 1 \leq j \leq l$$

Then for $1 \leq i \leq m$,

$$\sum Y_i = \sum_{j=1}^{n}(a_{ij}+b_{ij}) + \sum_{j=1}^{l}(c_{ij}+d_{ij})$$
$$= \sum_{j=1}^{n}(2mn+1) + \sum_{j=1}^{l}(4mn+2lm+1)$$
$$= (n+l)[2m(n+l)+1], \text{ for all } i \in [1, m]$$

Hence, the sum set partition of Y, $\sum(Y) = (\sum Y_1, \sum Y_2, \ldots, \sum Y_m)$ forms an arithmetic progression with common difference 0. Therefore, Y is m- balanced with all its subsets are $2(l+n)$-sets.

3 Main Results

Let $G = (V, E)$ be any (p, q) graph and the disjoint union of t copies of G can be denoted by tG. Then tG, $t \geq 1$ is a disconnected graph with vertex set $V(tG) = \{u_{ij} : 1 \leq i \leq t, 1 \leq j \leq p\}$ and the edge set $E(tG) = \{e_{ij} = u_{ij}u_{i(j+1)} : 1 \leq i \leq t, 1 \leq j \leq q\}$. Hence $|V(tG)| = tp$, $|E(tG)| = tq$. The next theorem presents a necessary condition for the graph $tG, t \geq 2$ to be G-super magic.

Theorem 3.1.*Let G be a (p, q) graph. If the graph tG, $t \geq 1$ is G-super magic, then any one of the following holds*

(i) p and q are of same parity

(ii) $t \equiv 1(mod 2)$ and p, q are of opposite parity

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and hence the magic constant is $\left(\frac{p+q}{2}\right)[t(p+q)+1]$

Proof. Suppose the graph tG, $t \geq 1$ is G-super magic. Then there is a total labeling $f : V(tG) \cup E(tG) \to \{1, 2, 3, \ldots, t(p+q)\}$ and for each graph G, there exists a positive integer c such that

$$\sum_{u_{ij} \in V(tG)} f(u_{ij}) + \sum_{e_{ij} \in E(tG)} f(e_{ij}) = c.$$

Thus for all t, $\sum_{u_{ij} \in V(tG)} f(u_{ij}) + \sum_{e_{ij} \in E(tG)} f(e_{ij}) = tc.$

$1 + 2 + 3 + \cdots + tp + (tp+1) + (tp+2) + \cdots + (tp+tq) = tc.$

$t(p+q)^2 + (p+q) = 2c$ …(1).

(1) holds only when either $p + q$ is even (or) $p + q$ is odd and $t \equiv 1 (mod\ 2)$

Therefore, (1) holds only when either (i) p and q are of same parity (or) (ii) $t \equiv 1 (mod\ 2)$ and p, q are of opposite parity and hence from (1), the magic constant is $\left(\frac{p+q}{2}\right)[t(p+q)+1]$.

The following theorems present the sufficient condition for the graph tG, $t \geq 1$ to be G-super magic.

Theorem 3.2. *Let G be a (p, q) graph with p and q are of same parity. Then the graph tG, $t \geq 1$ is G-super magic.*

Proof. Since$_{ij}$ p and q are of same parity, then we have $q = p + 2k$, $0 \leq k \leq \left\lfloor \frac{p^2-3p}{4} \right\rfloor$ and therefore the edge set becomes $E(tG) = \{e_{ij} : 1 \leq i \leq t, 1 \leq j \leq p\}$ $\cup$ $\{e_{ij}' : 1 \leq i \leq t, 1 \leq j \leq 2k\}$

Let $Z = [1, (p+q)t]$ and the partition of Z such that $Z = \bigcup_{i=1}^{t} [(W_i \cup X_i)] \cup Y_k \cup Y_k'$, where $W_i = [p(i-1)+1, p(i-1)+p]$, $X_i = [pt + p(i-1) +1, pt + p(i-1) + p]$ and for $0 \leq k \leq \left\lfloor \frac{p^2-3p}{4} \right\rfloor$, $Y_k = [2pt + 1, (2p + t)k]$, $Y_k' = [2(p+k)t + 1, 2(p+k)t]$.

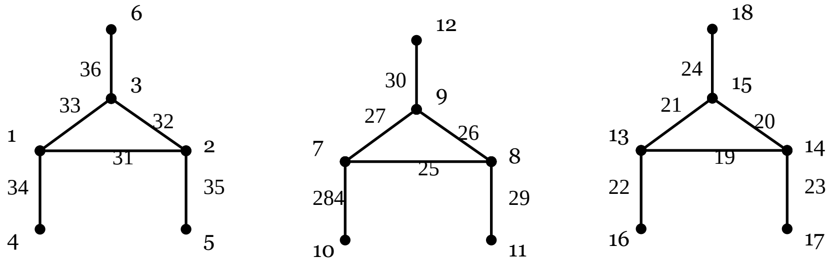
Now we define a total labeling f on tG as follows:

e_{ij} Label the vertices u_{ij}, $1 \leq i \leq t$, $1 \leq j \leq p$ with the$_k$ elements of W_i and label the edges e_{ij}, $1 \leq i \leq t$, $1 \leq j \leq p$ with the elements of X_i in decreasing order from left to right. Also label any k edges e_{ij}', $1 \leq i \leq t$, $1 \leq j \leq k$ with the elements of Y_k from left to right and lebel the remaing k edges e_{ij}', $1 \leq i \leq t$, $1 \leq j \leq k$ with the elements of Y_k' in decreasing order from right to left.

Let $n = p$, $m = t$ and $l = k$, then by the lemma 3.1 Z is t balanced with all its subsets are $2(p + k)$-sets and we have for $1 \leq i \leq t$ and $0 \leq k \leq \left\lfloor \frac{p^2-3p}{4} \right\rfloor$, $\sum Z_i = (p+k)[2t(p+k)+1]$.Hence $\sum(Z) = (\sum Z_1, \sum Z_2, \ldots, \sum Z_t)$ forms an arithmetic progression with common difference 0.

Therefore, the magic constant is given by $(p + k)[2t(p + k) + 1]$.

The following figures 1,2,3 illustrate the theorem 3.2.



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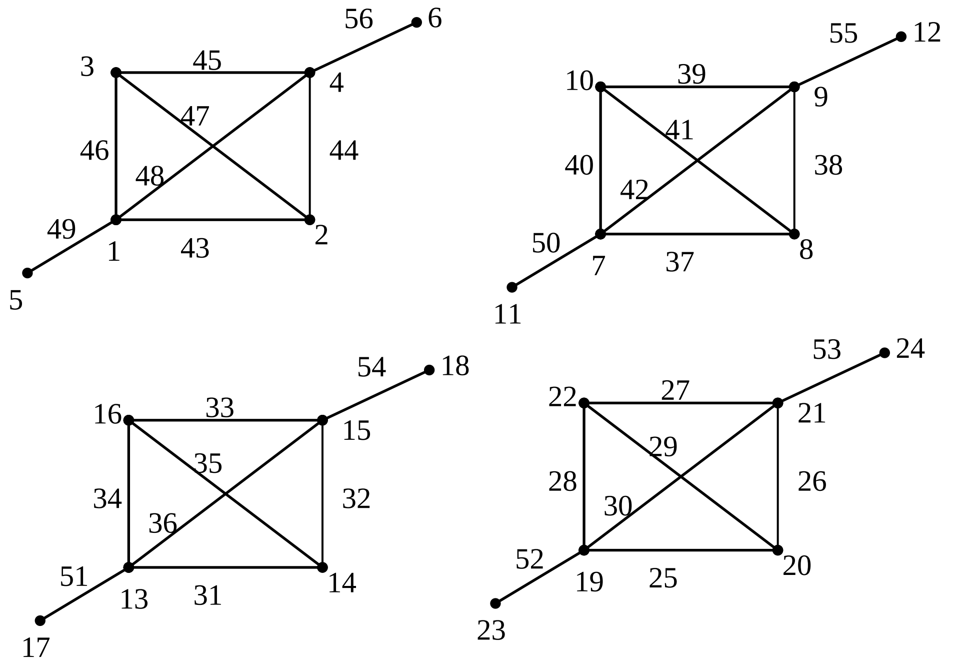
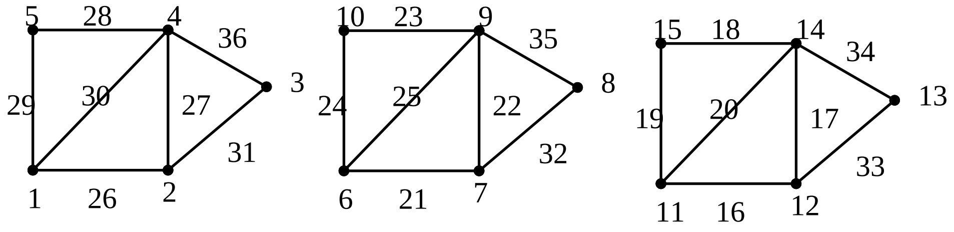
G(6, 6) *-supermagic total labeling and c=222*

Figure 1.



G(6, 8) *-supermagic total labeling and c=399*

Figure 2.



G(5, 7)*-supermagic total labeling and c=222*

Figure 3.

Theorem 3.3. *Let G be a (p,q) graph with p and q are of opposite parity and $q = p+1$, then for $t \equiv 1 (mod\ 2)$, the graph $tG, t \geq 1$ is G-super magic.*

Proof. We label the vertices and edges of the graph tG, $t \geq 1$ as follows:

$f(u_{ij}) = (i-1)p + j,\ 1 \leq i \leq t,\ 1 \leq j \leq p$

$$f(e_{ij}) = \begin{cases} (p+q)t-(i-1)q-j+1, & 1 \leq i \leq \left\lceil \frac{t}{2} \right\rceil, 1 \leq j \leq q \ and\ (i,j) \neq \left(i, \left(\left\lceil \frac{t}{2} \right\rceil - i + 2\right)\right); \\ (p+q)t-(i-1)q-j+1, & \left\lceil \frac{t}{2} \right\rceil + 1 \leq i \leq t, 1 \leq j \leq q \ and\ (i,j) \neq (i,1); \end{cases}$$

$$f\left(e_{i\left(\left\lfloor\frac{t}{2}\right\rfloor-1+2\right)}\right) = tp + qi\,, 1 \le i \le \left\lceil\frac{t}{2}\right\rceil$$

$$f(e_{i1}) = \frac{1}{2}(p+q)(t-1) + p(i+1) = 1, \left\lceil\frac{t}{2}\right\rceil + 1 \le i \le t$$

Let $wt(G^i)$ be the weight of the ith copy of the graph tG, $t \geq 1$

Then for $1 \le i \le \left\lceil\frac{t}{2}\right\rceil$,

$$wt(G^i) = \sum_{j=1}^{p} f(u_{ij}) + \sum_{j=1}^{q} f(e_{ij}) + f\left(e_{i\left(\left\lfloor\frac{t}{2}\right\rfloor-1+2\right)}\right) - f\left(e_{(t-i+1)1}\right)$$

$$= (i-1)p^2 + \frac{p(p+1)}{2} + tq(p+q) - (i-1)q^2 + q - \frac{q(q+1)}{2}$$
$$+ pt + qi - \frac{1}{2}(p+q)(t-1) - p(t-i+1+1) - 1$$

$$= (i-1)(p^2 - q^2 + p + q) - \tfrac{1}{2}(p+q)(t-1) + tp(p+q)$$

$$= \tfrac{(p+q)}{2}\,(2tq - t + 1)$$

$$wt(G^i) = \frac{(p+q)}{2}\,(t(p+q)+1)$$

Also, for $\left\lceil\frac{t}{2}\right\rceil + 1 \le i \le t$,

$$wt(G^i) = \sum_{j=1}^{p} f(u_{ij}) + \sum_{j=1}^{q} f(e_{ij}) + f\left(e_{(t+1-i)\left(i-\left\lceil\frac{t}{2}\right\rceil-i+2\right)}\right) - f(e_{i1})$$

$$= (i-1)p^2 + \frac{p(p+1)}{2} + tq(p+q) - (i-1)q^2 + q - \frac{q(q+1)}{2}$$
$$- pt - q(t+1-i) + \frac{1}{2}(p+q)(t-1) + p(i+1) + 1$$

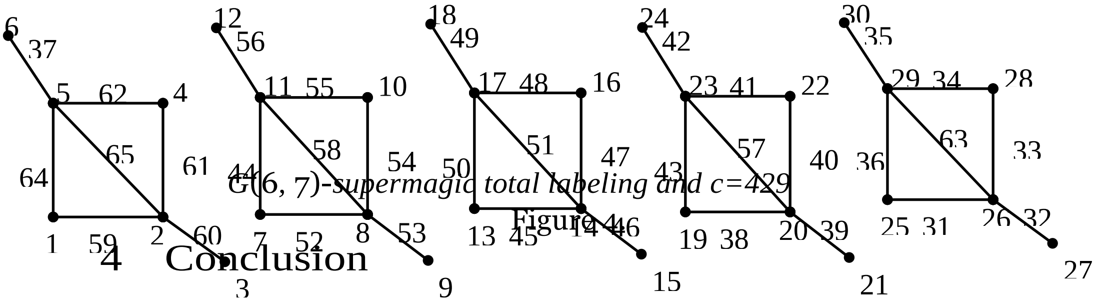
$$= \frac{(p+q)}{2}\,(2tq - t + 1)$$

$$wt(G^i) = \frac{(p+q)}{2}\,(t(p+q)+1)$$

Hence for each i, $1 \le i \le t$, the $wt(G^i)$ are the same.

Thus the graph $tG, t \equiv 1 (mod\ 2)$ and $t \geq 3$ is G- super magic with magic constant $\frac{(p+q)}{2}\,(t(p+q)+1)$.

The following figure 4 illustrates the theorem 3.3.



G(6, 7)-*supermagic total labeling and c=429*

Figure 4

4 Conclusion

In this paper we have shown that the graph tG, $t \geq 1$ has G-super magic

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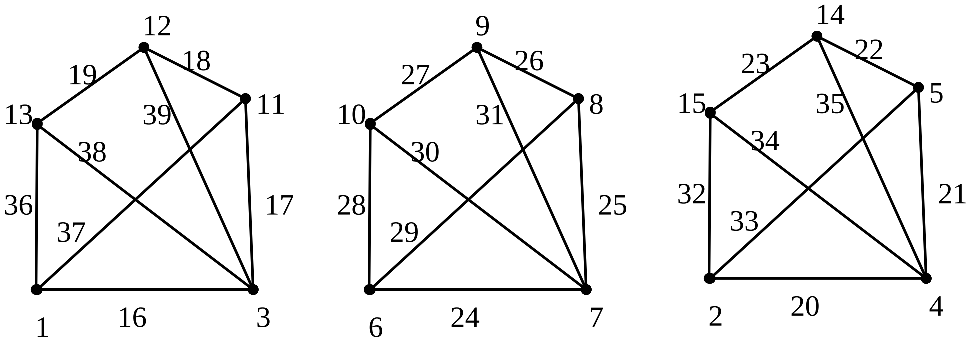
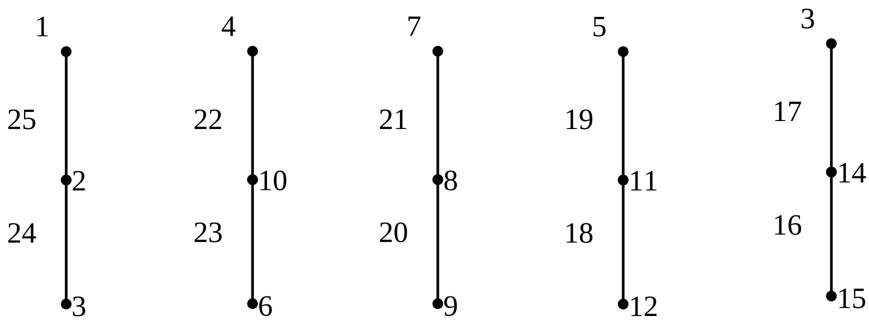
total labeling whenever (i) p and q are of same parity (ii) p and q are of opposite parity with $q = p + 1$ and $t \equiv 1(mod\ 2)$. Furthermore, we can find some examples(see Figures 4,5)of G-super magic total labeling for p and q are of opposite parity with $q \neq p + 1$ and $t \equiv 1(mod\ 2)$ and we have tried to find a G-super magic total labeling for p and p are of opposite parity with $q \neq p + 1$ and $t \equiv 1(mod\ 2)$, But we did not get success. Hence, we propose the following problem.

Problem 4.1. *For the graph tG determine if there is a G-super magic total labeling when p and q are of opposite parity with* $q \neq p + 1$ *and* $t \equiv 1(mod\ 2)$



G(5, 8)-*supermagic total labeling and c=260*
Figure 5.



G(3, 2)-*supermagic total labeling and c=65*
Figure 6.

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