COMMON FIXED POINT THEOREM FOR TWO MAPPINGS IN 2-BANACH SPACES

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Abstract
In this Paper a result for common fixed point theorem for two mappings is proved in 2-Banach Spaces, to add new rational mappings.

Keywords: **Banach Spaces,2-Banach Spaces, Fixed Point ,Common Fixed Point.**

1. INTRODUCTION

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear. therefore the fixed point methods specially Banach contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [24] described about the application of Banach's contraction Principle [2].

Browder [4] was the first mathematician to study non-expansive mappings. Meanwhile Brouwder [4] and Ghode [6] have independently proved a fixed point theorems for non-expansive mapping.

Many other mathematician viz;Datson [5] Goebel [6], Goebel and Zlotkienwicz and Bhagwan [23], Ahmad and Shakil [1], Shahzad and Udomene [24] have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk[15,16 and 17] gave the comprehensive survey concerning fixed point theorems for non-expansive mappings.

Ghalar[10] introduced the concept of 2-Banach Spaces. Recently Badshah and Gupta[3] also proved some results in 2-Banach Spaces. Yadava, Rajput and Bhardwaj [26], yadava , Rajput,Chaudhary and Bhardwaj [27] also worked for Banach and 2-Banach Spaces for non contraction mappings.

Recently Yadava,Rajput, Bhardwaj [27-30] Proved a results in 2- Banach Spaces for Non contraction Mappings as follows:

THEOREM 1: Let F be a mapping of Banach spaces X into itself. If F satisfies the following conditions:

(1.1) $F^2=I$

(1.2)

$$\|F(x)-F(y),a\| \le \alpha \frac{\|x-F(x),a\|\|y-F(x),a\|+\|y-F(y),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}$$

$$+\beta \frac{\|x-F(x),a\|\|y-F(y),a\|+\|y-F(x),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}$$

$$+\gamma \frac{\|x-F(x),a\|\|x-F(y),a\|+\|y-F(x),a\|\|y-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}+\delta\|x-y,a\|$$

then F has fixed point, further if

$\forall x,y\in X, x\neq y, \alpha \succ 0, 0\le \alpha,\beta,\gamma,\delta \prec 1 and 8\alpha+7\beta+8\gamma+4\delta \prec 8.$

$\beta+2\delta \prec 2$, then F has unique fixed point.

Now we are proving another common fixed point theorem in 2-Banach Spaces in which multiplication of two mappings is an identity mapping. That is GT=TG=L

DEFINITION (2.1.A): 2-Banach Spaces: In a Paper Gahler [10] defined a linear 2-normed Space to be pair (L, $\|.,.\|$) where L is a Linear space and $\|.,.\|$ is non negative, real valued function defined on L such that a,b,c $\in L$

(i) $\|a,b\|=0$ if and only if a and b are linearly dependent

(ii) $\|a,b\|=\|b,a\|$

(iii) $\|a,\beta b\|=\|\beta\|\|a,b\|, \beta\ is\ real$

(iv) $\|a,b+c\|\le\|a,b\|+\|a,c\|$

Hence $\|.,.\|$ is called a 2-norm.

DEFINITION (2.1.B): A sequence $\{X_n\}$ in a linear 2-normed Space L, is called a convergent sequence if there is, x $\in$ L, such that $\lim_{n\to\infty}\|x_n-x,y\|=0,\ for all y\in L.$

DEFINITION (2.1.C): A sequence $\{X_n\}$ in a linear 2-normed Space L, is called a Cauchy sequence if there exist y , z $\in$ L, such that y and z are linearly independent and $\lim_{m,n\to\infty}\|x_m-x_n,y\|=0$

DEFINITION (2.1.D): A Linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach Spaces.

3. Main Result

THEOREM 3: Let T and G be two non-expansive mappings of a 2-Banach space X into itself.T and G satisfy the following conditions:

(3.1) GT =TG=I, where I is identity mapping.

(3.2)

$$\|T(x)-G(y),a\| \le \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\|x-y\|}$$

$$+\beta \frac{\|x-T(x),a\|\|x-G(y),a\|+\|y-T(x),a\|\|y-G(y),a\|+\|x-y,a\|^2}{\|x-T(x),a\|+\|y-G(y),a\|+\|x-G(y),a\|+\|y-T(x),a\|+\|x-y,a\|}$$

$$+\gamma\|x-y,a\|+\delta\|x-T(x),a\|+\phi\|y-G(y),a\|+\eta\|x-G(y),a\|+\sigma\|y-T(x),a\|$$

For all x≠ y,

$$\alpha,\beta,\gamma,\delta,\eta \in [0,1[with \|x-T(x),a\|+\|y-G(y),a\|+\|x-G(y),a\|+\|y-T(x),a\|+\|x-y,a\| \ne 0$$

Then T and G have common Fixed point.

PROOF:

Taking $y=\frac{1}{2}\|(T+I)(x)\|, z=G(y), u=2y-z, Then \|z-x,a\|=\|G(y)-GT(x),a\|$

So by using (3.1) and (3.2), we get.

$$\|z-x,a\| \le \alpha \frac{\|y-G(y),a\|\|T(x)-G(T(x)),a\|}{\|y-T(x)\|}$$

$$+\beta \frac{\|y-G(y),a\|\|y-T(G(x)),a\|+\|T(x)-G(y),a\|\|T(x)-T(G(x)),a\|+\|y-T(x),a\|^2}{\|y-G(y),a\|+\|T(x)-T(G(x)),a\|+\|y-T(G(x)),a\|+\|T(x)-G(y),a\|+\|y-T(x),a\|}$$

$$+\gamma\|y-T(x),a\|+\delta\|y-G(y),a\|+\phi\|T(x)-T(G(x)),a\|+\eta\|y-T(G(x)),a\|+\sigma\|T(x)-G(y),a\|$$

$$\le \alpha \frac{\|y-G(y),a\|\|T(x)-x,a\|}{\frac{1}{2}\|x-T(x),a\|}$$

$$+\beta \frac{\|y-G(y),a\|\frac{1}{2}\|x-Tx,a\|+\|T(x)-y+y-G(y),a\|\|T(x)-x,a\|+\frac{1}{4}\|x-T(x),a\|^2}{\|y-T(x),a\|\phi+\|T(x)-x,a\|+\|y-x,a\|+\|y-T(x),a\|}$$

$$+\gamma\|y-T(x),a\|+\delta\|y-G(y),a\|+\phi\|T(x)-x,a\|+\eta\|y-x,a\|+\sigma\|T(x)-y+y-G(y),a\|$$

$$=2\alpha\|y-G(y),a\|+\frac{2}{5}\beta\left[\frac{3}{2}\|y-G(y),a\|+\frac{3}{4}\|T(x)-x,a\|\right]+\frac{1}{2}\gamma\|x-T(x),a\|$$

$$+\delta\|y-G(y),a\|+\phi\|T(x)-x,a\|]+\eta\|x-T(x),a\|+\sigma\|y-G(y),a\|]$$

$$\|z-x,a\| \le \|x-T(x),a\|\left[\frac{3\beta}{10}+\frac{\gamma}{2}+\delta+\phi+\eta\right]+\|y-G(y),a\|\left[2\alpha+\frac{6\beta}{10}+\delta+\sigma+\eta\right] \text{.......(3.3)}$$

Now we calculate $\|u-x,a\|=\|2y-z,a\|=\|T(x)-G(y),a\|$

$$\le \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\|x-y,a\|}$$

$$+\beta \frac{\|x-T(x),a\|\|x-G(y),a\|+\|y-T(x),a\|\|y-G(y),a\|+\|x-y,a\|^2}{\|x-T(x)\|+\|y-G(y)\|+\|x-G(y)\|+\|y-T(x)\|+\|x-y\|}$$

$$+\gamma\|x-y,a\|+\delta\|x-T(x),a\|+\phi\|y-G(y),a\|]+\eta\|x-G(y),a\|+\sigma\|y-T(x),a\|]$$

$$\leq \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\frac{1}{2}\|x-Tx,a\|}$$

$$+\beta \frac{\|x-T(x),a\|[\|x-y,a\|+\|y-Gy,a\|]+\frac{1}{2}\|x-T(x),a\|\|y-G(y),a\|+\frac{1}{4}\|x-T(x),a\|^2}{\frac{5}{2}\|x-T(x),a\|}$$

$$+\frac{1}{2}\gamma\|x-T(x),a\|+\delta\|x-T(x),a\|+\phi\|y-G(y),a\|]+\left[\eta[\frac{1}{2}\|x-T(x),a\|+\|G(y)-y,a\|]+\sigma\frac{1}{2}\|x-T(x),a\|\right]$$

$$\|u-x,a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{10}+\frac{\gamma}{2}+\delta+\sigma+\eta\right]+\|y-G(y),a\|\left[2\alpha+\frac{6\beta}{10}+\delta+\phi+\eta\right] ....(3.4)$$

Now $\|z-u,a\| \leq \|z-x,a\|+\|x-u,a\|$

$$\|z-u,a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{5}+\frac{\gamma}{1}+2\delta+2\eta+\phi+\sigma\right]+\|y-G(y),a\|\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta+\phi+\sigma\right] ....(3.5)$$

But $\|z-u,a\| = \|G(y)-2y+z\| = 2\|G(y)-y\|$........(3.6)

By(3.5) and (3.6)

$$2\|y-G(y),a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{5}+\frac{\gamma}{1}+2\delta+2\eta+\phi+\sigma\right]+\|y-G(y),a\|\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta+\phi+\sigma\right]$$

$$\|y-G(y),a\| \leq S\|x-T(x),a\|, where S = \frac{\left[\frac{3\beta}{5}+\frac{\gamma}{1}+2\delta+2\eta+\phi+\sigma\right]}{2-\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta+\phi+\sigma\right]} \prec 1, ...(3.7)$$

Because $20\alpha+9\beta+5\gamma+20\delta+20\eta+10\phi+10\sigma \prec 10$

Let F=$\frac{1}{2}[T+I]$,then for any x$\in X$

$$\|F^2(x)-F(x),a\| = \|F(Fx)-F(x),a\| = \|F(y)-y,a\| = \frac{1}{2}\|y-T(y),a\|$$

$$= \frac{1}{2}\|GT(y)-T(y),a\| \leq \frac{1}{2}\|G(y)-(y),a\|$$

Because T is non-expansive so $\|F^2(x)-F(x),a\| \leq \frac{S}{2}\|x-T(x),a\|$

By the definition of S, we claim that F(x) is a Cauchy sequence in X. Also by the completeness ,$F^n(x)$ convergence to some element (x_0)in X.

i.e $\lim_{n\to\infty} F^n(x) = x_0, \Rightarrow F(x_0) = x_0, hence T(x_0) = x_0$, That is x_0 is fixed point of T.

Again $\|F^2(x)-F(x),a\| \leq \frac{S}{2}\|x-T(x),a\| = \frac{S}{2}\|GT(x)-T(x),a\| \leq \frac{S}{2}\|x-G(x),a\|$

We Can conclude that $G(x_0)=x_0$. That is x_0 is fixed point of G. So $T(x_0)=G(x_0)=x_0$. So x_0 is common fixed point of T and G. The uniqueness part is trivial.

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