

COMMON FIXED POINT THEOREM FOR TWO MAPPINGS IN 2-BANACH SPACES

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Abstract

In this Paper a result for common fixed point theorem for two mappings is proved in 2-Banach Spaces, to add new rational mappings.

Keywords: Banach Spaces, 2-Banach Spaces, Fixed Point, Common Fixed Point.

1. INTRODUCTION

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear. therefore the fixed point methods specially Banach contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [24] described about the application of Banach's contraction Principle [2].

Browder [4] was the first mathematician to study non-expansive mappings. Meanwhile Brouwer [4] and Ghode [6] have independently proved a fixed point theorems for non-expansive mapping.

Many other mathematician viz; Dutton [5] Goebel [6], Goebel and Zlotkiewicz and Bhagwan [23], Ahmad and Shakil [1], Shahzad and Udomene [24] have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk [15, 16 and 17] gave the comprehensive survey concerning fixed point theorems for non-expansive mappings.

Ghalar [10] introduced the concept of 2-Banach Spaces. Recently Badshah and Gupta [3] also proved some results in 2-Banach Spaces. Yadava, Rajput and Bhardwaj [26], Yadava, Rajput, Chaudhary and Bhardwaj [27] also worked for Banach and 2-Banach Spaces for non contraction mappings.

Recently Yadava, Rajput, Bhardwaj [27-30] Proved a results in 2- Banach Spaces for Non contraction Mappings as follows:

THEOREM 1: Let F be a mapping of Banach spaces X into itself. If F satisfies the following conditions:

$$(1.1) F^2 = I$$

$$(1.2)$$

$$\begin{aligned} \|F(x)-F(y)\| &\leq \alpha \frac{\|x-F(x)\| \|y-F(x)\| + \|y-F(y)\| \|x-F(y)\|}{\|x-F(x)\| + \|y-F(x)\| + \|y-F(y)\| + \|x-F(y)\|} \\ &+ \beta \frac{\|x-F(x)\| \|y-F(y)\| + \|y-F(x)\| \|x-F(y)\|}{\|x-F(x)\| + \|y-F(x)\| + \|y-F(y)\| + \|x-F(y)\|} \\ &+ \gamma \frac{\|x-F(x)\| \|x-F(y)\| + \|y-F(x)\| \|y-F(y)\|}{\|x-F(x)\| + \|y-F(x)\| + \|y-F(y)\| + \|x-F(y)\|} + \delta \|x-y\| \end{aligned}$$

then F has fixed point, further if

$$\forall x, y \in X, x \neq y, \alpha > 0, 0 \leq \beta, \gamma, \delta < 1 \text{ and } 8\alpha + 7\beta + 8\gamma + 4\delta < 8$$

$$\beta + 2\delta < 2, \text{ then F has unique fixed point.}$$

Now we are proving another common fixed point theorem in 2-Banach Spaces in which multiplication of two mappings is an identity mapping. That is $GT=TG=L$

DEFINITION (2.1.A): 2-Banach Spaces: In a Paper Gähler [10] defined a linear 2-normed Space to be pair $(L, \|\cdot, \cdot\|)$ where L is a Linear space and $\|\cdot, \cdot\|$ is non negative, real valued function defined on L such that $a, b, c \in L$

- (i) $\|a, b\| = 0$ if and only if a and b are linearly dependent
- (ii) $\|a, b\| = \|b, a\|$
- (iii) $\|a, \beta b\| = |\beta| \|a, b\|, \beta$ is real
- (iv) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$

Hence $\|\cdot, \cdot\|$ is called a 2-norm.

DEFINITION (2.1.B): A sequence $\{X_n\}$ in a linear 2-normed Space L, is called a convergent sequence if there is, $x \in L$, such that $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0, \text{ forally } y \in L.$

DEFINITION (2.1.C): A sequence $\{X_n\}$ in a linear 2-normed Space L, is called a Cauchy sequence if there exist $y, z \in L$, such that y and z are linearly independent and $\lim_{m, n \rightarrow \infty} \|x_m - x_n, y\| = 0$

DEFINITION (2.1.D): A Linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach Spaces.

3. Main Result

THEOREM 3: Let T and G be two non-expansive mappings of a 2-Banach space X into itself. T and G satisfy the following conditions:

(3.1) $GT = TG = I$, where I is identity mapping.

(3.2)

$$\begin{aligned} \|T(x) - G(y), a\| &\leq \alpha \frac{\|x - T(x), a\| \|y - G(y), a\|}{\|x - y\|} \\ &+ \beta \frac{\|x - T(x), a\| \|x - G(y), a\| + \|y - T(x), a\| \|y - G(y), a\| + \|x - y, a\|^2}{\|x - T(x), a\| + \|y - G(y), a\| + \|x - G(y), a\| + \|y - T(x), a\| + \|x - y, a\|} \\ &+ \gamma \|x - y, a\| + \delta \|x - T(x), a\| + \phi \|y - G(y), a\| + \eta \|x - G(y), a\| + \sigma \|y - T(x), a\| \end{aligned}$$

For all $x \neq y$,

$$\alpha, \beta, \gamma, \delta, \eta \in [0,1] \text{ with } \|x - T(x), a\| + \|y - G(y), a\| + \|x - G(y), a\| + \|y - T(x), a\| + \|x - y, a\| \neq 0$$

Then T and G have common Fixed point.

PROOF:

Taking $y = \frac{1}{2} \|(T + I)(x)\|, z = G(y), u = 2y - z$, Then $\|z - x, a\| = \|G(y) - GT(x), a\|$

So by using (3.1) and (3.2), we get.

$$\begin{aligned} \|z - x, a\| &\leq \alpha \frac{\|y - G(y), a\| \|T(x) - G(T(x)), a\|}{\|y - T(x)\|} \\ &+ \beta \frac{\|y - G(y), a\| \|y - T(G(x)), a\| + \|T(x) - G(y), a\| \|T(x) - T(G(x)), a\| + \|y - T(x), a\|^2}{\|y - G(y), a\| + \|T(x) - T(G(x)), a\| + \|y - T(G(x)), a\| + \|T(x) - G(y), a\| + \|y - T(x), a\|} \\ &+ \gamma \|y - T(x), a\| + \delta \|y - G(y), a\| + \phi \|T(x) - T(G(x)), a\| + \eta \|y - T(G(x)), a\| + \sigma \|T(x) - G(y), a\| \\ &\leq \alpha \frac{\|y - G(y), a\| \|T(x) - x, a\|}{\frac{1}{2} \|x - T(x), a\|} \\ &+ \beta \frac{\|y - G(y), a\| \frac{1}{2} \|x - Tx, a\| + \|T(x) - y + y - G(y), a\| \|T(x) - x, a\| + \frac{1}{4} \|x - T(x), a\|^2}{\|y - T(x), a\| \phi + \|T(x) - x, a\| + \|y - x, a\| + \|y - T(x), a\|} \\ &+ \gamma \|y - T(x), a\| + \delta \|y - G(y), a\| + \phi \|T(x) - x, a\| + \eta \|y - x, a\| + \sigma \|T(x) - y + y - G(y), a\| \\ &= 2\alpha \|y - G(y), a\| + \frac{2}{5} \beta \left[\frac{3}{2} \|y - G(y), a\| + \frac{3}{4} \|T(x) - x, a\| \right] + \frac{1}{2} \gamma \|x - T(x), a\| \\ &+ \delta \|y - G(y), a\| + \phi \|T(x) - x, a\| + \eta \|x - T(x), a\| + \sigma \|y - G(y), a\| \end{aligned}$$

$$\|z - x, a\| \leq \|x - T(x), a\| \left[\frac{3\beta}{10} + \frac{\gamma}{2} + \delta + \phi + \eta \right] + \|y - G(y), a\| \left[2\alpha + \frac{6\beta}{10} + \delta + \sigma + \eta \right] \dots\dots(3.3)$$

Now we calculate $\|u - x, a\| = \|2y - z, a\| = \|T(x) - G(y), a\|$

$$\begin{aligned} &\leq \alpha \frac{\|x - T(x), a\| \|y - G(y), a\|}{\|x - y, a\|} \\ &+ \beta \frac{\|x - T(x), a\| \|x - G(y), a\| + \|y - T(x), a\| \|y - G(y), a\| + \|x - y, a\|^2}{\|x - T(x)\| + \|y - G(y)\| + \|x - G(y)\| + \|y - T(x)\| + \|x - y\|} \\ &+ \gamma \|x - y, a\| + \delta \|x - T(x), a\| + \phi \|y - G(y), a\| + \eta \|x - G(y), a\| + \sigma \|y - T(x), a\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha \frac{\|x - T(x), a\| \|y - G(y), a\|}{\frac{1}{2} \|x - Tx, a\|} \\ &+ \beta \frac{\|x - T(x), a\| [\|x - y, a\| + \|y - Gy, a\|] + \frac{1}{2} \|x - T(x), a\| \|y - G(y), a\| + \frac{1}{4} \|x - T(x), a\|^2}{\frac{5}{2} \|x - T(x), a\|} \\ &+ \frac{1}{2} \gamma \|x - T(x), a\| + \delta \|x - T(x), a\| + \phi \|y - G(y), a\| + \left[\eta \left[\frac{1}{2} \|x - T(x), a\| + \|G(y) - y, a\| \right] + \sigma \frac{1}{2} \|x - T(x), a\| \right] \end{aligned}$$

$$\|u - x, a\| \leq \|x - T(x), a\| \left[\frac{3\beta}{10} + \frac{\gamma}{2} + \delta + \sigma + \eta \right] + \|y - G(y), a\| \left[2\alpha + \frac{6\beta}{10} + \delta + \phi + \eta \right] \dots\dots(3.4)$$

Now $\|z - u, a\| \leq \|z - x, a\| + \|x - u, a\|$

$$\|z - u, a\| \leq \|x - T(x), a\| \left[\frac{3\beta}{5} + \frac{\gamma}{1} + 2\delta + 2\eta + \phi + \sigma \right] + \|y - G(y), a\| \left[4\alpha + \frac{6\beta}{5} + 2\delta + 2\eta + \phi + \sigma \right] \dots\dots(3.5)$$

But $\|z - u, a\| = \|G(y) - 2y + z\| = 2\|G(y) - y\| \dots\dots(3.6)$

By(3.5) and (3.6)

$$2\|y - G(y), a\| \leq \|x - T(x), a\| \left[\frac{3\beta}{5} + \frac{\gamma}{1} + 2\delta + 2\eta + \phi + \sigma \right] + \|y - G(y), a\| \left[4\alpha + \frac{6\beta}{5} + 2\delta + 2\eta + \phi + \sigma \right]$$

$$\|y - G(y), a\| \leq S \|x - T(x), a\|, \text{ where } S = \frac{\left[\frac{3\beta}{5} + \frac{\gamma}{1} + 2\delta + 2\eta + \phi + \sigma \right]}{2 - \left[4\alpha + \frac{6\beta}{5} + 2\delta + 2\eta + \phi + \sigma \right]} < 1, \dots\dots(3.7)$$

Because $20\alpha + 9\beta + 5\gamma + 20\delta + 20\eta + 10\phi + 10\sigma < 10$

Let $F = \frac{1}{2} [T + I]$, then for any $x \in X$

$$\begin{aligned} \|F^2(x) - F(x), a\| &= \|F(Fx) - F(x), a\| = \|F(y) - y, a\| = \frac{1}{2} \|y - T(y), a\| \\ &= \frac{1}{2} \|GT(y) - T(y), a\| \leq \frac{1}{2} \|G(y) - (y), a\| \end{aligned}$$

Because T is non-expansive so $\|F^2(x) - F(x), a\| \leq \frac{S}{2} \|x - T(x), a\|$

By the definition of S, we claim that F(x) is a Cauchy sequence in X. Also by the completeness, $F^n(x)$ convergence to some element (x_0) in X.

i.e $\lim_{n \rightarrow \infty} F^n(x) = x_0, \Rightarrow F(x_0) = x_0, \text{ hence } T(x_0) = x_0$. That is x_0 is fixed point of T.

Again $\|F^2(x) - F(x), a\| \leq \frac{S}{2} \|x - T(x), a\| = \frac{S}{2} \|GT(x) - T(x), a\| \leq \frac{S}{2} \|x - G(x), a\|$

We Can conclude that $G(x_0) = x_0$. That is x_0 is fixed point of G. So $T(x_0) = G(x_0) = x_0$. So x_0 is common fixed point of T and G. The uniqueness part is trivial.

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