A common fixed point theorem for two occasionally weakly compatible pairs in G-metric spaces using the common limit in the range property

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Abstract

In view of the fact that the fixed point theory provides an efficient tool in many fields of pure and applied sciences, we use the notion of Common limit in the range property to prove a common fixed point theorem for occasionally weakly compatible mappings.

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*CLRg*property.

INTRODUCTION

Inspired by the fact that the metric fixed point theory provides an efficient tool in many fields of pure and applied sciences, many authors investigated the possibility to generalize the notion of a metric space. In this direction, Gahler [1, 2] introduced the notion of a 2-metric space, while Dhage [3] introduced the concept of a D-metric space. Later on, Mustafa and Sims[4] showed that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, they introduced a new notion of a generalized metric space, called G-metric space. After then, many authors studied fixed and common fixed points in generalized metric spaces; see [4-15].

Here, we give preliminaries and basic definitions which are helpful in the sequel. First, we introduce the concepts of a G-metric and a G-metric space.

Definition 1.1[4] Let X be a nonempty set and $G: X \times X \times X \to [0, +\infty)$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0 \; if \; x = y = z$;

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2 A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Example 1.3 Let $X = \{2,3\}$ and $G: X \times X \times X \to [0, +\infty)$ be defined by $G(2, 2,2) = G(3, 3,3) = 0$, $G(2, 2,3) = G(2\ 3,2) = G(3, 2, 2) = 1$, $G(2, 3, 3) = G(3, 2, 3) = G(3, 3, 2) = 2$. It is easy to show that the function G satisfies all properties of Definition 1.1, but $G(x, x, y) \neq G(x, y, y)$ for all $x, y \in X$ with $x \neq y$. Therefore, G is not symmetric.

Definition 1.4 [4] Let(X, G) be a G-metric space, and let $\{x_n\}$ be sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x or $\{x_n\}$ G-converges to x.

Thus, $x_n \to x$ in a G-metric space (X, G) if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$.

Proposition 1.5[4]*Let*(X, G) *be a G-metric space. Then the following are equivalent:*

(1) $\{x_n\}$*is G- convergent to x;*
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$;
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$.

Definition 1.6 [4] Let(X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq k$; that is, $G(x_n, x_m, x_l) \to 0$as $n, m, l \to +\infty$.

Proposition 1.7 [4] *Let*(X, G) *be a G-metric space. Then the following are equivalent:*

(1) *The sequence* $\{x_n\}$ *is G-Cauchy;*
(2) *for every* $\varepsilon > 0$, *there is* $k \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$ *for all* $n, m \geq k$.

Proposition 1.8 [4] *Let*(X, G) *be a G-metric space. Then the function* $G(x, y, z)$*is jointly continuous in all three of its variables.*

Definition 1.9 [4] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.10 [4] Let(X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that

(i) if $G(x, y, z) = 0$, *then* $x = y = z$;
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;
(iii) $G(x, y, y) \leq 2G(y, x, x)$;
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;
(v) $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$;
(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

An interesting observation is that any G-metric space (X, G) induces a metric d_G on X given by

$d_G(x, y) = G(x, y, y) + G(y, x, x)$, for all $x, y \in X$.

Moreover, (X, G) is G-complete if and only if (X, d_G) is complete.

It was observed that in the symmetric case ((X, G) is symmetric), many fixed point theorems on G-metric spaces are particular cases of the existing fixed point theorems in metric spaces. This allows us to readily transport many results from the metric spaces into the G-metric spaces.

On the other hand, by reasoning on the properties of the mappings, the practise of coining weaker forms of commutativity to ensure the existence of a common fixed point for self-mappings on metric spaces is still on. To read more in this direction, we refer to [16] and the references therein. Here, for our further use, we recall only the two fundamental notions of 'occasionallyweakly compatible mappings' and 'common limit in the range property'; see also [17, 18]

In 1976,Jungck [26] introduced the notion of occasionally weakly compatible mappings as follows.

Definition 1.11 Let S and T be two self-mappings of a metric space *(X,d)*. Then the pair *(S, T)* is said to be weakly compatible if they commute at their coincidence points, that is if $Su = Tu$ for some $u \in X$, then $TSu = STu$.

Definition 1.12 Let S and T be two mappings from a metric space *(X, d)* into itself. Then, the mappings are said tobe occasionally weakly compatible(owc) if and if there is a point $z \in X$ which is a coincidence point of S and T at which S and T commute. i.e., there exists a point $z \in X$ such that $Sz=Tz$ and $STz=TSz$.

In 2002, Amari and El Moutawakil [20] introduced a new concept of the property E.A in metric spaces to generalize the concept of noncompatible mappings. Then, they proved some common fixed point theorems.

Definition 1.12 Let S and T be two self-mappings of a metric space (X,d). Then the pair (S, T) is said to satisfy the property E.A if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X$.

Definition 1.13A pair (A, S) of self mappings of a metric space (X,d) is said to satisfy the (CLR_S) property with respect to mapping S, if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$,where$t \in S(X)$.

Define the (CLR_{ST}) property (with respect to mappings of S and T) as follows.

Definition 1.14 Satisfy the common limit range property with respect to mappings S and T (briefly, (CLR_{ST}) property), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

forsome $t \in S(X) \cap T(X)$.

Example 1.15 Let $X = [0, +\infty)$. Define $S, T: X \to X$ by $Sx = \frac{3}{4}x$ and $Tx = \frac{x}{4}$ for all $x \in X$.Consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$ in X. Clearly, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = 0 \in X$, and so S and T satisfy the property E.A.

Example 1.16 Let $X = [2, +\infty)$. Define $S, T: X \to X$ by $Sx = 2x + 1$ and $Tx = x + 1$ for all $x \in X$.Suppose that the property E.A holds. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X$. It follows that $\lim_{n\to\infty} x_n = \frac{t-1}{2}$ and $\lim_{n\to\infty} x_n = t - 1$ and so, by Definition 1.12, $t = 1$ but $t \notin X$. Therefore, S and T do not satisfy the property E.A

In conclusion of this preliminary section, we consider the following set:

Let Φ denote the set of all the functions $\varphi: [0, +\infty) \to [0, +\infty)$ such that:

(1) φ is non-decreasing;
(2) $\lim_{n\to\infty} \varphi^n(r) = 0$ for all $r \in [0, +\infty)$.

If $\varphi \in \Phi$, then it is an easy matter to show that $\varphi(0) = 0$ and $\varphi(r) < r$ for all $r \in [0, +\infty)$; see Matkowski [21].

In this paper, by merging the above notions, we prove a common fixed point theorem for two pairs of occasionally weakly compatible mappings in a G-metric space.

Lemma 1.17:[26] Let *X* be a set, *S* and *T* be occasionally weakly compatible(owc) self maps on *X*. If *S* and *T* have a unique point of coincidence *w*=*Sx*=*Tx* for *x* ∈ *X*, then *w* is the unique common fixed point of *S* and *T*.

2Main result:

The following is the main result of this section.

Theorem 2.1*Let (X,G) be a G-metric space and* $P, Q, A, B, S, T: X \to X$ *be six self-mappings such that :*

2.1.1 *for all* $x, y \in X$,

$$G(Px, Qy, Qy) \leq \varphi(max\{G(ABx, STy, STy), G(ABx, Qy, Qy), G(STy, Qy, Qy)\}), where\ \varphi \in \Phi;$$

2.1.2 *(P, AB) and (Q, ST) satisfies* $CLR_{(AB)(ST)}$ *property*

Then the pairs (P, AB) and (Q, ST) have a coincidence point. Moreover, P, Q, AB, and ST have a unique common fixed point in X provided that both pairs (P, AB) and (Q, ST) are occasionally weakly compatible. Further if (A, B) , (S, T), (A, P) and (S, Q) are commuting maps then A, B, S, T, P and Q have a unique common fixed point.

Proof. Since *(P, AB)* and *(Q, ST)* satisfies $CLR_{(AB)(ST)}$ property, there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} STy_n = t$ where $t \in AB(X) \cap ST(X)$. Since $t \in AB(X)$, there exists a point $u \in X$ such that $ABu = t$. We shall show that *Pu*=*ABu*. Suppose that $Pu \neq ABu$, then from (2.1.1), with $x = u, y = y_n$, we get

$$G(Pu, Qy_n, Qy_n) \leq \varphi(max\{G(ABu, STy_n, STy_n), G(ABu, Qy_n, Qy_n), G(STy_n, Qy_n, Qy_n)\})$$

Taking the limit as $n \to +\infty$, by the property of φ, we get

$$G(Pu, t, t) \leq \varphi(max\{G(t, t, t), G(t, t, t), G(t, t, t)\}) = \varphi(0) = 0,$$

which implies $Pu = ABu = t$. Therefore, *u*is a coincidence point of the pair *(P, AB)*.

Also $t \in ST(X)$, there exists a point $v \in X$ such that $STv = t$.

Next, we show that $Qv = STv = t$. Let, on the contrary $Qv \neq STv$, then from (2.1.1) and using the fact that $\varphi(r) < r$, with $x = u,\ y = v$ we have

$$G(t, Qv, Qv) = G(Pu, Qv, Qv) \leq \varphi(max\{G(ABu, STv, STv), G(ABu, Qv, Qv), G(STv, Qv, Qv)\})$$

$$= \varphi(max\{G(t,t,t), G(t, Qv, Qv), G(t, Qv, Qv)\})$$

$$= \varphi(max\{0, G(t, Qv, Qv), G(t, Qv, Qv)\})$$

$$= \varphi\big(G(t, Qv, Qv)\big)$$

$$< G(t, Qv, Qv)$$

which contradicts. Therefore $Qv = STv = t$, which shows that *v*is a coincidence point of the pair *(Q,ST)*

Since the pair (*P*, *AB*) are occasionally weakly compatible so by definition there exists a point $u \in X$ such that *Pu=ABu* and *P(AB)u=(AB)Pu*

Since the pair (*Q*, *ST*) are occasionally weakly compatible so by definition there exists a point $v \in X$ such that *Qv=STv* and *Q(ST)v=(ST)Qv*

Moreover, if there is another point z such that *Pz= ABz*, then, using **(2.1.1)** it follows that *Pz=ABz=Qv=STv*, or *Pu=Pz* and *w=Pu=ABu* is unique point of coincidence of *P* and *AB*. ByLemma 1.17, w is the unique common fixed point of *P* and *AB*. i.e., *w=Pw=ABw*. Similarly there is a unique point $z \in X$ such that *z=Qz=STz*.

Uniqueness: Suppose that $w \neq z$. Using inequality **(2.1.1)** with *x=w*, *y=z*, we get

$$G(w, z, z) = G(Pw, Qz, Qz)$$

$$\leq \varphi(max\{G(ABw, STz, STz), G(ABw, Qz, Qz), G(STz, Qz, Qz)\})$$

$$= \varphi(max\{G(w, z, z), G(w, z, z), G(z, z, z)\})$$

$$= \varphi\big(G(w, z, z)\big)$$

$$< G(w, z, z)$$

That is a contradiction and so must be *w=z*. Therefore, P, Q, AB and ST have a unique common fixed point.

Finally we need to show that *z* is a common fixed point of *A*, *B*, *P*, *Q*, *S* and *T*

Since (*A*, *B*), (*A*, *P*) are commutative

Az=*A*(*ABz*)=*A*(*BAz*)=(*AB*)*Az*;

Az=*APz*=*PAz*

Bz=*B*(*ABz*)=(*BA*)*Bz*=(*AB*)*Bz*;

Bz=*BPz*=*PBz*

which shows that Az, Bz are common fixed point of (AB, P) yielding then by

$Az=z=Bz=Pz=ABz$ in the view of uniqueness of common fixed point of the pairs (P, AB)

Similarly using the commutativity of (S,T) and (S,Q) it can be shown that, $Sz=z=Tz=Qz=STzSz=z=Tz=Qz=Az=Bz=Pz$.

Therefore, $Sz=z=Tz=Qz=Az=Bz=Pz$.

which shows that z is a common fixed point of A, B, P, Q, S and T.

If we assume $A=S$ and $T=B=I_x$ in above Theorem 2.1, we deduce the following result involving three self-mappings

Corollary 2.2 Let (X, G) be a G-metric space and $P, Q, A: X \to X$ be three mappings such that:

2.2.1 *for all* $x, y \in X$,

$$G(Px, Qy, Qy) \le \varphi(max\{G(Ax, Ay, Ay), G(Ax, Qy, Qy), G(Ay, Qy, Qy)\}), where\ \varphi \in \Phi;$$

2.2.2 *(P, A) and (Q, A) satisfies* $CLR_{(A)}$ *property*

Then the pairs (P, A) and (Q, A) have a coincidence point. Moreover, P, Q and A have a unique common fixed point in X provided that both pairs (P, A) and (Q, A) are occasionally weakly compatible.

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